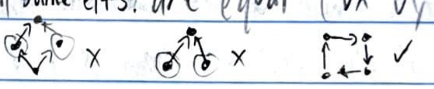
- Kuratowski's pairs: $(a,b) := \{\{a,b\},\{a\}\}$ (encoding ordered pairs w/ sets)
- ordered triples: $(a,b,c) := (a,(b,c))$
- def: a function is a set f s.t. every element of f is an ordered pair and if $(a,b),(a,b') \in f$, then $b = b'$
- von Neumann numbers: $0 = \emptyset$, $1 = \{0\} = \{\emptyset\}$, $2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$, ..., $n+1 = n \cup \{n\}$
- def: A universe of set theory is a collection $\mathcal{U}$ of objects together w/ a binary relation $\in$ (i.e. it's a directed graph). Members of $\mathcal{U}$ (vertices of graph) are called sets
- def: a free variable is a variable that doesn't appear after a quantifier in a formula
- def: replacing some free vars w/ members of $\mathcal{U}$, we get a formula w/ parameters
- def: a sentence is a formula with no free variables, is either true or false
  - if φ is a true sentence in $\mathcal{U}$, we write $\mathcal{U} \models \varphi$

① • Axiom of Extensionality (Ext): sets w/ same elts. are equal ($\forall x \forall y ((x \subseteq y \wedge y \subseteq x) \to x = y)$)
  - $x \subseteq y$: $\forall z (z \in x \Rightarrow z \in y)$ 
- def: a class is a different name for a formula w/ one free variable (and possibly w/ params). Given a formula $\varphi(x)$, the corresponding class is $\mathcal{C} = \{x : \varphi(x)\}$ ← this is not a set
- ex: $\mathcal{U} = \{x : x = x\}$ - the class of all sets
  $\emptyset = \{x : x \neq x\}$ - the empty class
  $\{x : \exists y \forall z (z \in x \Leftrightarrow z = y)\}$ - the class of all 1-element sets
  $\{x : \exists a \exists b (x = (a,b))\}$ - the class of all ordered pairs
- def: a class $\mathcal{C}$ is (represented by) a set if there is a set a in $\mathcal{U}$ s.t. $\mathcal{C} = \{x : x \in a\}$
  Note: by Ext, such an a must be unique. If there is no set, $\mathcal{C}$ is a proper class
  Russel's paradox: $R = \{x : x \notin x\}$ - this is a proper class

② • Empty Set Axiom (Empty): the empty class $\emptyset = \{x : x \neq x\}$ is a set
  - $\exists x \forall y (y \notin x)$

③ • Pairing Axiom (Pair): for any a,b in $\mathcal{U}$, the class $\{a,b\} := \{x : x = a \vee x = b\}$ is a set
  - $\forall a \forall b \exists x \forall y (y \in x \Leftrightarrow (y = a \vee y = b))$
  $\{a,a\} = \{a\} = \{x : x \in a\}$ is a set by Pair

④ • Axiom of Union (Union): for any S in $\mathcal{U}$, the class $\bigcup S = \{x : \underbrace{\exists y \in S (x \in y)}_{\exists y (y \in S \wedge x \in y)}\}$ is a set
  - $\exists z \forall x \exists y (x \in y \wedge y \in S \to x \in z)$
  If A, B are sets in $\mathcal{U}$, $\bigcup \{A, B\} = A \cup B$ is a set.

(5) • Power set Axiom (Pow): for all sets a, the class $\mathcal{P}(a) = \{x : x \subseteq a\}$ is a set

(6) • Axiom Schema of Comprehension (Comp): If $\mathcal{C}$ is a class and $\mathcal{C} \subseteq a$ for some set a, then $\mathcal{C}$ is a set ("small" classes are sets)

• ex. $X \times Y = \{(x,y) : x \in X, y \in Y\} = \{z : \exists x \exists y \, (x \in X \wedge y \in Y \wedge z = (x,y))\}$ is a set since $(x,y) = \{\{x\},\{x,y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y))$, so $X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$

• def: a (binary) relation is a set of binary pairs. for a relation R, let
$\mathrm{dom}(R) := \{x : \exists y \, ((x,y) \in R)\}$. $\mathrm{dom}(R)$ is a set since $\mathrm{dom}(R) \subseteq \bigcup\bigcup R$.
$\mathrm{ran}(R) := \{y : \exists x \, ((x,y) \in R)\}$ is also a set since $\mathrm{ran}(R) \subseteq \bigcup\bigcup R$.
the image of of a set A under R is $R[A] := \{y : \exists x \in A \, ((x,y) \in R)\}$ is a set

• def a function is a relation f st for every $x \in \mathcal{U}$, there is at most one y w/ $(x,y) \in f$
 ▹ $\forall x \forall y \forall z \, (((x,y) \in f \wedge (x,z) \in f) \rightarrow y = z)$
 ▹ given sets X, Y, we say f is a function from X to Y ($f : X \to Y$) if $\mathrm{dom}(f) = X$, $\mathrm{ran}(f) \subseteq Y$
 ◦ the class of all functions from X to Y is denoted ${}^{X}Y$
  → if X, Y are sets, then ${}^{X}Y$ is a set

◦ def a set X is inductive if $\emptyset \in X$ and $\forall x \in X$, $x \cup \{x\} \in X$

(7) • Axiom of Infinity (Inf): There exists an inductive set.

• thm: There exists an inclusion-minimal, unique, inductive set ω
 ▹ for any inductive set X, $\omega \subseteq X$

Pf: Uniqueness holds by Ext. Consider the class $\omega = \{x : x \in X$ for any inductive set $X\}$.
By Inf, $\exists$ an inductive set X and by Comp, $\omega \subseteq X$ is a set.
By def, $\emptyset \in \omega$ as $\emptyset$ is in all inductive sets. Let $y \in \omega$, then y is in every inductive set, hence $y \cup \{y\}$ is in every ind. set, so $y \in \omega$. □

• def: the elements of ω are called "natural numbers" (in $\mathcal{U}$)
 ★ to prove things abt natural numbers, we must use the fact that ω/natural numbers are the inclusion minimal set.

• ex: for all $n \in \omega$, either $n = 0$ or $0 \in n$.
Pf: let X be the set of natural numbers n st $n = 0$ or $0 \in n$. Obviously, $X \subseteq \omega$.
We WTS X is inductive, so that $\omega \subseteq X$. Note $\emptyset = 0 \in X$. Next, let $n \in X$.
If $n = 0$, then $\emptyset \cup \{0\} = \{0\} \ni 0$ so $\{0\} \in X$. If instead $0 \in n$, then $n \cup \{n\} \supseteq n \ni 0$, so $n \cup \{n\} \in X$, hence X is inductive. □

- def: a class function Φ is a formula w/ 2 free vars (possibly w/ parameters) $\varphi(x,y)$, s.t. $\forall x \forall y \forall z\, ((\varphi(x,y) \wedge \varphi(x,z)) \to y = z)$
  - for $x \in dom(\Phi)$, $\Phi(x)$ denotes the unique $y \in ran(\Phi)$ st. $\varphi(x,y)$.
- examples: - $x \mapsto x$ (id), given by formula $x = y$
  - $x \mapsto \{x\}$ has formula $\forall z\, (z = y \Leftrightarrow z = x)$
  - $x \mapsto \mathcal{P}(x)$ (formula is exercise)
- def: for a class C and class fn. Φ, C's image by Φ is the class $\Phi[C] = \{\Phi(x) : x \in C \cap dom(\Phi)\}$

(8) - **Axiom Schema of Replacement (Rep)**: The image of a set under a class fn. is a set.
- def: let $V_0 = \emptyset$, $V_1 = \mathcal{P}(V_0)$, ..., $V_{n+1} = \mathcal{P}(V_n)$, $V_\omega = \bigcup\{V_n : n \in \omega\}$.
  If we can show $n \mapsto V_n$ is a class fn (hard part), then by Rep, $\{V_n : n \in \omega\}$ is a set, then V_ω is a set by Union.
- Dedekind's Formalization of Recursion: $y = V_n \Leftrightarrow \exists w_0 \exists w_1 \dots \exists w_n \wedge (\forall i < n\, (w_{i+1} = \mathcal{P}(w_i))) \wedge w_n = y$.
  the class fn for $n \mapsto V_n$: we want a formula that says "$x \in \omega$ and $y = V_x$", this is:
  $x \in \omega \wedge \exists W$ st. 1) W is a fn, $dom(W) = x+1 = x \cup \{x\} = \{0,1,2,\dots,x\}$
  2) $W(0) = \emptyset$ 3) $\forall i < x$ ($= i \in x$) $W(i+1) = \mathcal{P}(W(i))$ 4) $W(x) = y$
  → how to write? 1) $\exists y\, (x \in y \wedge y = \omega)$, y is the inclusion-min inductive set
  2) $\forall y$ (y is inductive $\to x \in y$) 3) treat $x \in \omega$ as a formula w/ param (ω is param)
- Prop: there is no injective class fn from a proper class to a set.
  Pf: Suppose $\Phi : \mathcal{C} \to X$ is injective class fn from a class $\mathcal{C}$ to set X.
  $ran(\Phi) \subseteq X$ is a set by Comp.. Then $\Phi^{-1} : ran(\Phi) \to \mathcal{C}$ is a class fn (by injectivity), so $\Phi^{-1}(ran(\Phi)) = \mathcal{C}$ is a set by Rep. □
- example of proper classes: $R := \{x : x \in x\}$ (Russell), $\mathcal{U}$ (if not, $R \subseteq \mathcal{U}$ would be a set by Comp)
- def: let $\mathcal{C}$ be a class, $\prec$ a class relation on $\mathcal{C}$, i.e. a formula $\varphi(x,y)$ (w/ params), st. $\forall x \forall y\, (\varphi(x,y) \to x, y \in \mathcal{C})$. We write $x \prec y$ to mean $\varphi(x,y)$
- exercise: If S is a set and $\prec$ a class relation on S, then it is a set relation, i.e., $\exists R \subseteq S \times S$ set st. $x \prec y \Leftrightarrow (x,y) \in R$.
- def: $\prec$ has strict partial order if it is
  (P1): irreflexive: $\forall x\, \neg(x \prec x)$ (P2): transitive: $\forall x \forall y \forall z\, (x \prec y \prec z \to x \prec z)$

- def: a partial order is total / linear if $\forall x, y \in \mathcal{C}$, $x \prec y \vee x = y \vee x \succ y$
- def: a linear ordering on a class $\mathcal{C}$ is a well ordering if:

(W1): every nonempty subset $S \subseteq \mathcal{C}$ has a $\prec$-least element, i.e.,
$\exists x \in S \ \forall y \in S \ (x \preceq y)$

(W2): for each $x \in \mathcal{C}$, its initial segment $S_x(\mathcal{C}) := \{y \in \mathcal{C} : y \prec x\}$ is a set.

▸ If $\mathcal{C}$ is a set, W2 holds automatically by Comp.

- Prop: If $\prec$ is a well ordering on a class $\mathcal{C}$, then every nonempty subclass $\mathcal{D} \subseteq \mathcal{C}$ has a $\prec$-least element

Pf: let $x \in \mathcal{D}$. Then $S := \mathcal{D} \cap S_x(\mathcal{C})$ is a set by (W2) and Comp. If $S = \emptyset$, then x is the $\prec$-least elt of $\mathcal{D}$. Else, by (W1), there is a $\prec$-least elt of S, which is also $\prec$-least elt of $\mathcal{D}$. □

- def: a set A is transitive if every elt of A is a subset of A:

▸ $\forall x \, (x \in A \to x \subseteq A)$ or $\forall x \forall y \, (y \in x \in A \to y \in A)$

- def: an ordinal is a set α s.t.

(O1): α is transitive (O2): α is well ordered by $\in$

▸ the class of all ordinals is denoted Ord.

- exercise: Ord is a class
- note: $\emptyset \in \mathrm{Ord}$.
- note: if $\alpha \in \mathrm{Ord}$, then $\alpha \notin \alpha$ (else $\in$ is not irreflexive)
- def: let $\alpha \in \mathrm{Ord}$. The successor of α is $\alpha+1 := \alpha \cup \{\alpha\}$
- Prop: $\alpha \in \mathrm{Ord} \Rightarrow \alpha+1 \in \mathrm{Ord}$

Pf: (O1): let $\beta \in \alpha+1$. If $\beta \in \alpha$, then $\beta \subseteq \alpha \subseteq \alpha+1$. Else $\beta = \alpha \subseteq \alpha+1$. So $\alpha+1$ transitive.
(O2): take any nonempty subset $S \subseteq \alpha+1$. If $S \cap \alpha \neq \emptyset$, then the $\in$-least elt of $S \cap \alpha$ is also a $\in$-least elt of S. Else, $S = \{\alpha\}$, and α is the $\in$-least elt (and only elt) of S. □

- Corollary: $\omega \subseteq \mathrm{Ord}$

Pf: The set $\omega \cap \mathrm{Ord}$ is inductive (has $\emptyset$ and closed under successor). As ω is a minimum inductive set, $\omega = \omega \cap \mathrm{Ord}$. □

- exercise: ω is an ordinal.
- note: every elt of an ordinal is an ordinal

- thm: the class Ord is well-ordered by $\in$
- ex: an ordinal w/ one elt? $\alpha = \{x\}$

  by transitivity, $x \subseteq \{x\}$ so $x = \emptyset$ or $x = \{x\}$, second not possible as $x \in x$ breaks transitivity. so $x = \emptyset$, $\alpha = \{\emptyset\} = 1$.
- ex: an ordinal w/ two elts? $\alpha = \{x, y\}$, $x \neq y$

  $\in$ is a strict linear order, so WLOG, $x \in y$ $(x \notin x, y \notin y)$. Also $y \notin x$.
  By transitivity, $x \subseteq \alpha = \{x, y\} \Rightarrow x = \emptyset$. But then $y \subseteq \alpha = \{\emptyset, y\} \Rightarrow y = \{\emptyset\}$.
  so $\alpha = \{\emptyset, \{\emptyset\}\} = 2$.
- Lemma: $\in$ is a strict partial order on Ord

  Pf: Irreflexive: If $\alpha \in$ Ord, $\alpha \notin \alpha$. ✓

  Transitivity: Sps α, β, γ are ordinals s.t. $\alpha \in \beta \in \gamma$. As γ is a transitive set, $\alpha \in \gamma$. ✓ □

  * by partial order, we can write $\beta < \alpha$ to mean $\beta \in \alpha$.

  ★ so $\alpha = \{\beta \in \text{Ord} : \beta < \alpha\}$ ★
- def: let $\mathcal{C}$ be a class w/ strict partial order $\prec$. A subclass D is downwards-closed if $\forall x \in D\ \forall y \in \mathcal{C}\ (y \prec x \rightarrow y \in D)$.
- Lemma: If $\alpha \in$ Ord and $D \subseteq \alpha$ is downwards closed, then either $D \in \alpha$ or $D = \alpha$. In particular D is an ordinal.

  Pf: If $D = \alpha$ we're done, so assume $D \neq \alpha$, that is, $\alpha - D \neq \emptyset$. Let β be the least elt of $\alpha - D$, which exists by well ordering of α.
  D is downwards closed $\Rightarrow \forall \gamma \in D\ (\beta > \gamma)$. Thus $D \subseteq \beta$. Alternatively if $\gamma \in \beta$, i.e. $\gamma < \beta$, then $\gamma \in D$ by minimality of β. Thus $\beta \subseteq D$, so $D = \beta \in \alpha$. □
- clear: downward closed sets of ordinals are ordinals.
- Lemma: If $\alpha, \beta \in$ Ord, then $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$.

  Pf: let $\gamma = \alpha \cap \beta$. Then γ is a downward closed subset of both α and β.

  Case 1: $\gamma = \alpha$, $\gamma = \beta$. then $\gamma = \alpha = \beta$, done

  Case 2: $\gamma = \alpha$, $\gamma \in \beta$. then $\alpha \in \beta \Rightarrow \alpha < \beta$, done

  Case 3: $\gamma \in \beta$, $\gamma = \alpha$, then $\beta < \alpha$, done

  Case 4: $\gamma \in \alpha$, $\gamma \in \beta$, but this is not possible as then $\gamma \in \alpha \cap \beta = \gamma$. □

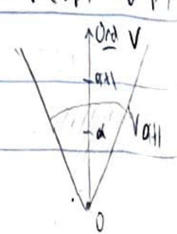
Pf (Well ordering): We already know $<$ or $\in$ is a strict linear order on $\underline{Ord}$.
(W1): let $S \subseteq \underline{Ord}$ be nonempty. take any $\alpha \in S$. If α is least elt of S, we're done. Else, $\alpha \cap S = \{\beta \in S ; \beta < \alpha\} \subseteq \alpha$ is nonempty, and thus has a least elt β as α is an ordinal. Also β is least elt in S.
(W2): $S_\alpha(\underline{Ord}) = \{b \in \text{Ord}. \beta < \alpha\} = \alpha$ is a set. □

- Corollary: $\underline{Ord}$ is a proper class
  Pf: If $\underline{Ord}$ is a set, it would be a transitive set well-ordered by $\in$, i.e., $\underline{Ord}$ is an ordinal. But then $\underline{Ord} \in \underline{Ord}$, a contradiction (ordinals can't be their own elt). □
- def: for a fn f and a set S, let $f \upharpoonright S$ denote the restriction of f to S: $f \upharpoonright S := f \cap (S \times \mathcal{U})$
- def: let Extend be a class function. A set function f is Extend-inductive if
  (I1): the domain of f is an ordinal (I2): for all $\beta < \alpha$, $f(\beta) = \text{Extend}(f \upharpoonright \beta)$ $\{\gamma : \gamma < \beta\}$
- lemma: let $\alpha \in \text{Ord}$, Extend a class fn. If $f, g : \alpha \to \mathcal{U}$ are Extend-inductive, then $f = g$.
  Pf: Suppose not, let $\beta < \alpha$ be the least st. $f(\beta) \neq g(\beta)$. By choice of β, $f(\gamma) = g(\gamma)$ for all $\gamma < \beta$, i.e. $f \upharpoonright_\beta = g \upharpoonright_\beta$. Thus $f(\beta) = \text{Extend}(f \upharpoonright \beta) = \text{Extend}(g \upharpoonright \beta) = g(\beta)$, ※. □

7 • thm (transfinite recursion): let $\alpha \in \underline{Ord}$, Extend a class fn. Suppose that for every Extend-inductive fn $g : \gamma \to \mathcal{U}$ st. $\gamma < \alpha$, $g \in \text{dom}(\text{Extend})$. Then $\exists!$ Extend-inductive $f : \alpha \to \mathcal{U}$.
- ex: $f_0 : 0 \to \mathcal{U}$, $f_0 = \emptyset$ $f_1 : 1 \to \mathcal{U}$, $f_1(0) = \mathcal{E}(f_0) = \mathcal{E}(\emptyset)$ or $f_1 = \{(0, \mathcal{E}(f_0))\}$
  $f_2 = \{(0, \mathcal{E}(f_0)), (1, \mathcal{E}(f_1))\} = f_1 \cup \{(1, \mathcal{E}(f_1))\}$, $f_{n+1} = f_n \cup \{(n, \mathcal{E}(f_n))\}$ ← successor
  $f_\omega = \{(0, \mathcal{E}(f_0)), (1, \mathcal{E}(f_1)), (2, \mathcal{E}(f_2)), \dots\} = \bigcup_{n \in \omega} f_n$ ← limit ordinal
  ★ $\omega \notin \bigcup_{\beta < \omega+1} \beta$, so $\bigcup_{\beta < \omega+1} f_\beta \neq f_{\omega+1}$
- thm (transfinite recursion for classes): let $\mathcal{E}$ be a class fn. Suppose for all $\mathcal{E}$-ind. functions g, $g \in \text{dom}(\mathcal{E})$. Then $\exists$ class fn. $\Phi : \underline{Ord} \to \mathcal{U}$ st. $\forall \alpha \in \text{Ord}\, (\Phi(\alpha) = \mathcal{E}(\Phi \upharpoonright \alpha))$.
  Pf: Define $\Phi(\alpha) = y$ iff: $\exists f\,(f$ is $\mathcal{E}$-ind. fn (it's unique!) w/ $\text{dom}(f) = \alpha + 1$ and $f(\alpha) = y)$ □
- def: define a class fn $\underline{Ord} \to \mathcal{U} : \alpha \mapsto V_\alpha$ as follows: $V_\alpha := \bigcup\{\mathcal{P}(V_\gamma) : \gamma < \alpha\}$
  ▷ formally, we apply transfinite rec. (classes) for class fn $\mathcal{E}$ by $\mathcal{E}(f) = \bigcup\{\mathcal{P}(f(x)) ; x \in \text{dom}(f)\}$
- Note: If $\beta \leq \alpha$, $V_\beta \subseteq V_\alpha$. Also, $V_{\beta+1} = \bigcup_{\gamma < \beta+1} \mathcal{P}(V_\gamma) = \mathcal{P}(V_\beta)$ (since $\mathcal{P}(V_\gamma) \subseteq \mathcal{P}(V_\beta)\ \forall \gamma < \beta$)
  Conversely, if α is a limit, $V_\alpha = \bigcup_{\gamma < \alpha} \mathcal{P}(V_\gamma) = \bigcup_{\gamma < \alpha} V_{\gamma+1} = \bigcup_{\gamma < \alpha} V_\gamma$

  together: $V_\alpha = \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \mathcal{P}(V_\beta) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\gamma < \alpha} V_\gamma & \text{if } \alpha \text{ is limit} \end{cases}$

  $\alpha \in V_{\alpha+1}$



(a)

- def: the von Neumann Universe is the class $V := \bigcup\{V_\alpha : \alpha \in Ord\} = \{x : \exists\alpha(\alpha \in Ord \wedge x \in V_\alpha)\}$
- Axiom of Foundation (or Regularity) (AF): $\mathcal{U} = V$
  $\triangleright \forall x \, \exists\alpha(\alpha \in Ord \wedge x \in V_\alpha)$
- def: $ZF^- = Ext + Empty + Pair + Union + Pow + Comp + Inf + Rep$. $ZF = ZF^- + AF$
- thm: Assuming ZF^-, AF is equivalent to $\forall x(x \neq \emptyset \to \exists y \in x(y \cap x = \emptyset))$
- def: the rank of x is the least ordinal α st. $x \in V_\alpha$ (rank is class fn)
- note: 1) If $x \in V$, then rank(x) is a successor 2) if $y \in x \in V$, then rank$(y) <$ rank(x)
  3) If $\alpha \in Ord$, then rank$(\alpha) = \alpha + 1$ (Hint: consider least ordinal for which it fails)
- Prop: $x \in V$ iff $\forall y \in x (y \in V)$ $(x \subseteq V)$
  Pf: $(\Rightarrow)$: given by (2) in note
  $(\Leftarrow)$: Suppose all elts of x are in V. By Rep, $\{\text{rank}(y) : y \in x\}$ is a set. Hence $\beta = \bigcup\{\text{rank}(y) : y \in x\}$ is a set of ordinals and is downwards closed, so β is an ordinal. for each $y \in x$, $y \in V_{\text{rank}(y)} \subseteq V_\beta$ (as rank$(y) \leq \beta$), which implies $x \subseteq V_\beta$, so $x \in V_{\beta+1}$ and $x \in V$. □

  Pf (AF equivalence): $(\Rightarrow)$: let $x \neq \emptyset$, $x \in V = \mathcal{U}$. Consider $\{\text{rank}(y) : y \in x\}$, which is a set by Rep. It is a nonempty set of ordinals, so it has a least elt α. Take any elt $y \in x$ w/ rank$(y) = \alpha$. then every elt of x has rank $\geq \alpha$, but every elt of y has rank $< \alpha$, so $x \cap y = \emptyset$.
  $(\Leftarrow)$: let $x \in \mathcal{U}$ (WTS $x \in V$). First assume x is transitive (i.e. all elts of x are subsets of x). Suppose FSOC that $x \notin V$. By Prop, not all elts of x are in V, so consider nonempty $\tilde{x} := x \setminus V \subseteq x$ which is a set (by Comp). So $\exists y \in \tilde{x} \subseteq x$ st $y \cap \tilde{x} = \emptyset$. But $y \subseteq x$ by transitivity of x. So $y \subseteq x \setminus \tilde{x} = x \cap V$, thus every elt of y is in V. So $y \in V \Rightarrow y \notin \tilde{x}$ ↯
  Now take any set x. We prove the following lemma: for every x, $\exists y$ transitive st. $x \subseteq y$. If we can find such a y, then by the previous case, $y \in V$, so $y \subseteq V$ (by Prop), so $x \subseteq y \subseteq V$, thus $x \in V$ by Prop again, as desired.
  Define a function $\omega \to \mathcal{U} : n \mapsto y_n$ recursively by $y_0 := x$, $y_{n+1} = \bigcup y_n$ for all $n \in \omega$, and $y = \bigcup\{y_n : n \in \omega\}$. Note $x = y_0 \subseteq y$, secondly, we claim y is transitive. Take $z \in y$, so $z \in y_n$ for some $n \in \omega$, and since $y_{n+1} = \bigcup y_n$, $z \subseteq y_{n+1} \subseteq y$. □
- Corollary: $\forall x, \exists y$ transitive st. $x \in y$. Pf: Take z from lemma, let $y := z \cup \{x\}$. □

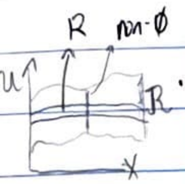
- Note: the construction in Lemma give the smallest transitive set containing X. So we call this set the transitive closure of X.
- Prop. $AF \Rightarrow \neg\exists x\,(x \in x)$ (no set is its own elt)

  Pf1: If $x \in x \in V$, then $\text{rank}(x) < \text{rank}(x)$ ↯ $\square$

  Pf2: $AF \Leftrightarrow$ every non $\emptyset$ set has an elt disjoint from itself. Say $x \in x$, consider $\{x\}$ (nonempty), it only has one elt x, but $x \cap \{x\} = \{x\} \neq \emptyset$. ↯ $\square$
- Prop (in ZF): there is no seq of sets w/ $x_0 \ni x_1 \ni x_2 \ni \dots$. Or, there is no fn $f: \omega \to \mathcal{U}$ st. $\forall n \in \omega$, $f(n+1) \in f(n)$.

  Pf: Suppose there was such a function f. Then $\forall n \in \omega$, $\text{rank}(f(n+1)) < \text{rank}(f(n))$. But the ordinals are well ordered, but the set (by Rep) $\{\text{rank}(f(n)) : n \in \omega\}$ has no least elt and is non-$\emptyset$ set of ordinals. $\square$

  Also, $\text{ran}(f) = \{f(n) : n \in \omega\}$ is non-$\emptyset$ set that intersects each of its elts: $f(n) \cap \text{ran}(f) \ni f(n+1)$ $\square$



- Prop (in ZF): let X be a set, $\mathcal{R} \subseteq X \times \mathcal{U}$ be a class relation st. $\forall x \in X\ \exists y\ ((x,y) \in \mathcal{R})$. Then $\exists$ a set relation $R \subseteq \mathcal{R}$ st. $\forall x \in X, \exists y\ ((x,y) \in R)$

  Pf: Define $(x,y) \in R \Leftrightarrow (x,y) \in \mathcal{R} \wedge \forall z\,((x,z) \in \mathcal{R} \to \text{rank}(y) \leq \text{rank}(z))$. If $x \in X$, $\alpha = \min(\text{rank}(y) : (x,y) \in \mathcal{R})$, then $\{y : (x,y) \in R\} = \{y : (x,y) \in \mathcal{R}\} \cap V_\alpha$, so it's a nonempty set. This implies R is a set (Rep). $\square$

back to ZF$^-$
- Thm: Let $(A, <)$ be a well ordered set. Then $\exists!$ a ordinal that's order-isomorphic to $(A, <)$, moreover, the isomorphism is also unique.
  - this α is called the order type (or just type) of $(A, <)$, denoted $\text{type}(A, <)$

  Pf: Uniqueness: HW.

  Existence: Suppose $(A, <)$ is well ordered, st. there is no order presrving bijection b/w A and any ordinal. We want to define a class fn $\Phi : \text{Ord} \to A$ recursively by $\Phi(\beta) = \min A \setminus \Phi[\beta]$. Consider the class fn $\mathcal{E}$ (extend) by $\mathcal{E}(f) = y \Leftrightarrow$ f is function, $\text{ran}(f) \subsetneq A$, and $y = \min A \setminus \text{ran}(f)$. For all β, $\Phi(\beta) = \mathcal{E}(\Phi \restriction \beta) = \min(A \setminus \text{ran}(\Phi \restriction \beta)) = \min(A \setminus \Phi[\beta])$.

  Claim: Let $\alpha \in \text{Ord}$, $f : \alpha \to \mathcal{U}$ is $\mathcal{E}$-ind. fn. Then a) $\text{ran}(f) \subseteq A$. b) f is injective c) f strictly increasing. d) $\text{ran}(f) \neq A$ e) $f \in \text{dom}(\mathcal{E})$

Pf (a): f is $\mathcal{E}$-ind, for all $\beta < \alpha$, $f(\beta) = \mathcal{E}(f \restriction \beta) = \min A \setminus f[\beta] \in A$.

Pf (b): Take $\gamma < \beta < \alpha$ [WTS $f(\gamma) \neq f(\beta)$], $f(\beta) = \min A \setminus f[\beta]$ and $f(\beta) \in A \setminus f[\beta] \not\ni f(\gamma)$ so $f(\beta) \neq f(\gamma)$.

Pf (c): take $\gamma < \beta < \alpha$, so $\gamma \subsetneq \beta$, so $f[\gamma] \subseteq f[\beta]$ then $A \setminus f[\gamma] \supseteq A \setminus f[\beta]$, hence $f(\gamma) = \min(A \setminus f[\gamma]) \leq \min(A \setminus f[\beta]) = f(\beta)$, and f is inj, so $f(\gamma) < f(\beta)$.

Pf (d): Else, f is a bijection hence an order isomorphism b/w α and $(A, \prec)$, which we assumed didn't exist.

Pf (e): by def of $\mathcal{E}$.

By transfinite recursion thm (classes), $\exists$ $\mathcal{E}$-ind class fn $\Phi : \underline{Ord} \to A$, i.e., a class fn defined on $\underline{Ord}$ s.t. $\forall \alpha \in \underline{Ord}$, $\Phi \restriction \alpha$ is $\mathcal{E}$-ind. Then by Claim, $\operatorname{ran} \Phi \subseteq A$ and Φ is injective. But now we have an injection from proper class to a set. $\lightning$ $\square$

- Similarly, every well ordered proper class $(\mathcal{C}, \prec)$ is isomorphic to $\underline{Ord}$, $\Phi : \underline{Ord} \to \mathcal{C}$ given by $\Phi(\beta) := \min \mathcal{C} \setminus \{\Phi(\gamma) : \gamma < \beta\} = \min \mathcal{C} \setminus \Phi[\beta]$
- def: a choice fn for a set X is $f : X \to \bigcup X$ s.t. $\forall x \in X\ (f(x) \in x)$
- ex: $X = \{a\}$, $a \neq \emptyset$, since a nonempty, a has an elt b. By Pair, (a, b) is a set. By Pair again $\{(a,b)\}$ is a set and this is a choice fn for X.
- def: A set X is finite if there is a bijection $n \to X$ for some $n \in \omega$, else infinite
- Thm (finite choice) (ZF^-): If X is a finite set and $\emptyset \notin X$, then X has a choice fn.

Pf: Let $x : n \to X$ be a bijection (so $X = \{x(0), x(1), \dots, x(n-1)\}$).

Base case: ($n = 0$): $X = \emptyset$, and $\emptyset$ is a choice fn for X.

Inductive step: Suppose the statement is true when X is in bijection w/ $n \in \omega$, and let $x : (n+1) \to X$ be a bijection, then $X = \{x(0), \dots, x(n-1), x(n)\}$. Let $X' = \{x(i) : i < n\} = x[n]$. Then $x \restriction n$ is a bijection from n to X', so by ind. hyp., $\exists$ a choice fn $f' : X' \to \bigcup X'$ for X'. $x(n)$ is nonempty so it has an elt, say $y \in x(n)$. Define a choice fn $f : X \to \bigcup X$ by $f := f' \cup \{(x(n), y)\}$. $\square$

- ex: $X = \mathcal{P}(\omega) \setminus \{\emptyset\}$: has a choice fn (and is infinite). $f(A) := \min(A)$, $A \subseteq \omega$
- Axiom of Choice (AC): If $\emptyset \notin X$, then X has a choice fn
- $ZFC = ZF + AC$, $ZFC^- := ZF^- + AC$ (no foundation)

- thm (ZFC$^-$): If X is infinite, there is an injection $\omega \to X$

Pf: X is infinite means there does not exist a bijection $n \to X$ ($n \in \omega$). $f: \omega \to X$ recursively defined as $f(n) := $ any elt of $X \setminus f[n]$. By AC, there is a choice fn $\underline{\text{choice}}: P(X) \setminus \{\emptyset\} \to X$, define $f n := \underline{\text{choice}}(X \setminus f[n])$. we apply transfinite recursion thm to extension fn: (X, choice are parameters) $\mathcal{E}(f) = x \iff f$ is a function, $\operatorname{ran}(f) \subsetneq X$, and $x = \underline{\text{choice}}(X \setminus \operatorname{ran}(f))$ □

- $\underline{\text{thm}}$ (Well Ordering) (ZF$^-$): TFAE: (1) AC (2) for every set A, there is a well ordering on A (3) for every set A, there exists an ordinal α and bijection $\alpha \to A$

$\underline{\text{Pf}}$: (2)$\Rightarrow$(3): every well ordered set is order isomorphic to an ordinal

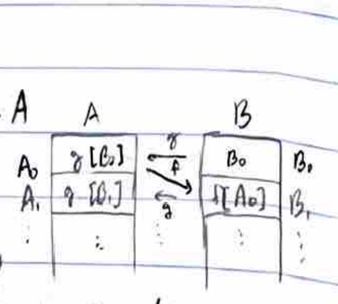
(3)$\Rightarrow$(2): let $f: A \to \alpha$ be a bijection, $\alpha \in \underline{\text{Ord}}$. define a well ordering on A by $a < b \iff f(a) < f(b)$

(2)$\Rightarrow$(1): Take a set X, $\emptyset \notin X$. Let $\prec$ be a well ordering on $\bigcup X$ and define a choice fn $f: X \to \bigcup X$ by $f(x) = \min_{\prec} x$.

(1)$\Rightarrow$(3): Let A be a set. Suppose for contradiction that A is not in bijection with any ordinal. Define a class fn $\Phi: \underline{\text{Ord}} \to A$ recursively as follows: AC take a choice fn $\underline{\text{choice}}: P(A) \setminus \{\emptyset\} \to A$ and set $\Phi(\beta) := \underline{\text{choice}}(A \setminus \Phi(\beta))$. this is valid b/c $\Phi[\beta] \neq A$ for all β. So we get an injective class fn $\Phi: \text{Ord} \to A$ which is a contradiction as we can't inject proper class into a set. □

- $\underline{\text{def}}$: Let A, B be sets. Write $A \precsim B$ if there is an injection $A \to B$. We write $A \approx B$ if there is a bijection $A \to B$ (we say A, B are equinumerous)
  - A is finite if $A \approx n$ for some $n \in \omega$, infinite else
  - A is countable if $A \precsim \omega$

- thm (Cantor-Schröder-Bernstein) (ZF$^-$): $A \approx B \iff A \precsim B$ and $B \precsim A$



$\underline{\text{Pf}}$: ($\Rightarrow$): bijection is injective and inverse is injective also

($\Leftarrow$): let $f: A \to B$, $g: B \to A$ be injections. Let $B_0 = B \setminus f[A]$, $A_0 = g[B_0]$, $B_1 = f[A_0]$, $A_1 = g[B_1]$, ... By construction, $B_0, B_1, \ldots$ are disjoint, $A_0, A_1, \ldots$ are disjoint, $g \restriction B_n$ is a bijection. Hence letting $B' = \bigcup_n B_n$, $A' = \bigcup A_n$, $g \restriction B'$ is a bijection from $B' \to A'$. Let $B'' = B \setminus B'$, $A'' = A \setminus A'$. Note $f \restriction A''$ is a bijection from A'' to B''. Then defining $h: A \to B$ as $h(a) = g^{-1}(a)$ if $a \in A'$, and $h(a) = f(a)$ if $a \in A''$, is a bijection. □

thm (Cantor) (ZF⁻): for any X, $X \precsim \mathcal{P}(X)$ but $\mathcal{P}(X) \not\precsim X$. Moreover, there is a class function W st. $\forall X, \forall f: X \to \mathcal{P}(X)$, $W(X,f) \in \mathcal{P}(X) \setminus ran(f)$.

Pf: $X \precsim \mathcal{P}(X)$ by $x \mapsto \{x\}$. Define for any $f: X \to \mathcal{P}(X)$, $W(X,f) := \{x \in X : x \notin f(x)\} \subseteq X$. Suppose for contradiction $W(X,f) \in ran(f)$, so $W(X,f) = f(x)$ for some x. If $x \in W(X,f)$, then by def of W, $x \notin f(x) = W(X,f)$ ↯. But if $x \notin W(X,f)$, then $x \in W(X,f)$ by def of W, ↯. So $W(X,f) \notin ran(f)$. So there cannot be a surjection (hence bijection) of $X \to \mathcal{P}(X)$ □

- thm (Hartogs) (ZF⁻): for every X, there is an ordinal α st. $\alpha \not\precsim X$.

Pf: Let $S := \{\alpha \in \underline{Ord} : \alpha \precsim X\}$. We claim X is a set (so $S \neq \underline{Ord}$). If we have an injection $f: \alpha \to X$, $\alpha \in \underline{Ord}$, we get a well ordering on $ran(f) \subseteq X$ of order type α. Conversely, if $<$ is a well ordering on a subset $Y \subseteq X$, then we have an injection $type(<) \to X$ (a bijection $type(<) \to Y$).

So, $S = \{type(<) : <$ is a well-ordering on a subset of $X\}$. Note $W = \{< : <$ is a well ordering on subsets of $X\} \subseteq \mathcal{P}(X \times X)$ (well ordering is a set of ordered pairs). So W is a set by Pow, Comp, then S is a set by Rep, so there is an ordinal in $\underline{Ord}$ (a proper class) but not in S (a set) □

  - ex. S is the least ordinal which doesn't inject into X

- thm (ZF⁻): TFAE: (1) AC (2) for every set A, B, $A \precsim B$ or $B \precsim A$.

Pf: (1)⇒(2): Assume AC, let A, B sets. There exists ordinals $\alpha \approx A$, $\beta \approx B$ (AC !). WLOG, $\alpha \precsim \beta$ ($\alpha \subseteq \beta$), then $A \approx \alpha \precsim \beta \approx B$ so $A \precsim B$.

(2)⇒(1): Assume (2). We will show every set can be well ordered (equiv to AC). Let X be a set. Then $\exists \alpha \in \underline{Ord}$ st. $\alpha \not\precsim X$ by Hartog's. Then $X \precsim \alpha$ by (2), an injection $f: X \to \alpha$. Putting $x < y \Leftrightarrow f(x) < f(y)$ gives a well ordering on X. □

- def: say X be well-orderable if there is a well ordering on X ⇔ equinumerous w/ an ordinal
- def: If X is well-orderable, it's cardinality is $|X| := \min(\alpha \in \underline{Ord} : \alpha \approx X)$
  - AC ⇔ every set has a cardinality
- def: a cardinal is an ordinal κ s.t. there is no smaller ordinal $\gamma < \kappa$ and $\gamma \approx \kappa$
  - X well orderable, then $|X|$ is a cardinal
  - $\alpha \in \underline{Ord}$ is a cardinal iff $|\alpha| = \alpha$ iff no injection to any smaller ordinal

• **Prop**: If κ is a cardinal, γ is an ordinal w/ $\kappa \precsim \gamma$, then $\kappa \le \gamma$.

PF: let $f: \kappa \to \gamma$ be an injection. If $\gamma < \kappa$ ($\gamma \subsetneq \kappa$) then there is an injection $\gamma \to \kappa$, so by C-S-B, $\kappa \approx \gamma$. But we have a contradiction as $\gamma < \kappa$ but κ is an ordinal. □

PF (w/o CSB): let $f: \kappa \to \gamma$ injective, then $ran(f) \subseteq \gamma$ is well ordered. Then there is a strictly increasing bijection from $\delta = type(ran(f), <)$ and $ran(f)$ ($\delta \to ran(f) \subseteq \gamma$). Then $\delta \le \gamma$ by HW2. So $\kappa \approx \delta$ by $\kappa \xrightarrow{f} ran\, f \xrightarrow{h^{-1}} \delta$ □

∘ Assuming AC: $A \precsim B \Leftrightarrow |A| \le |B|$, $A \approx B \Leftrightarrow |A| = |B|$.

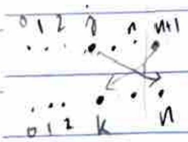
• For any well-orderable sets A, B, $A \precsim B \Leftrightarrow |A| \le |B|$, $A \approx B \Leftrightarrow |A| = |B|$

• thm: every natural # is a cardinal (if $n < m < \omega$, then $n \not\approx m$)

PF: Since $n+1 \le m$, it's enough to show $n+1 \not\precsim n$.

Base case ($n=0$): there are no (injective) fns from $1 = \{\emptyset\}$ to $0 = \emptyset$.

Inductive step: Suppose $n+1 \not\precsim n$. Suppose for contradiction that $f: n+2 \to n+1$ is an injection. If $f(n+1) = n$, then $f \restriction n+1 = f \restriction \{0, \dots n\}$ is an injection from $n+1 \to n$, a contradiction. Next if $n \notin ran(f)$, then $f \restriction (n+1)$ is still an injection $n+1 \to n$, a contradiction. Lastly, then for some $k < n$, $f(n+1) = k$ and for some $j < n+1$, $f(j) = n$.



Swap these values / map $j \to k$; and restrict to $(n+1)$ to produce an injection. □

• Corollary: ω is a cardinal

PF: If $\omega \approx n$ for some $n < \omega$, then $(n+1) \subseteq \omega \approx n \Rightarrow n+1 \precsim n$, contradiction. □

• **Prop**: An infinite cardinal is a limit ordinal (not a successor)

• By Hartogs, for every cardinal, there is a strictly larger cardinal

• def: for a cardinal κ, κ^+ = the successor (cardinal) of κ = $\min(\lambda \in Card, \lambda > \kappa)$
$= \{\alpha \in Ord : |\alpha| \le \kappa\}$

• note: Card is a proper class

• note: Since $Card \setminus \omega$ = {infinite cardinals} is a well-ordered proper class, we have an order-preserving, bijective class fn $Ord \to Card \setminus \omega$ denoted $\aleph$, $\alpha \mapsto \aleph_\alpha$, $(\aleph_\alpha)^+ = \aleph_{\alpha+1}$ for all $\alpha \in Ord$, $\aleph_0 = \omega$

• assuming AC (every set has a cardinality): Continuum Hyp: $|\mathcal{P}(\omega)| = \aleph_1$

without AC: If $\omega \precsim A \precsim \mathcal{P}(\omega)$, then either $A \approx \omega$ or $A \approx \mathcal{P}(\omega)$

- def: If $(X, <_X)$, $(Y, <_Y)$ are linear orders, $(X,<_X) \oplus (Y,<_Y)$ is a linear order that looks like $\underbrace{\quad}_{X,<_X}\underbrace{\quad}_{Y,<_Y}$. To disjointify, the underlying set is $X \sqcup Y := (X\times\{0\}) \cup (Y\times\{1\})$
  - if $<_X, <_Y$ are well orderings, $<_X \oplus <_Y$ is also a well ordering (also $\otimes$)
- def: $(X,<_X)\otimes(Y,<_Y)$ is an ordering of $X\times Y$:
- def: for $\alpha, \beta \in \underline{Ord}$, $\alpha+\beta = \text{type}(\alpha\oplus\beta)$, $\alpha\beta = \text{type}(\alpha\otimes\beta)$
- note: $\alpha\cdot 2 = \alpha+\alpha$ because $\alpha\otimes 2 = \alpha\oplus\alpha$ $\underbrace{\quad}_{\alpha\times\{0\}}\underbrace{\quad}_{\alpha\times\{1\}}$

$\omega+n = \text{type}(\ldots)$ $\quad \omega+\omega = \text{type}(\ldots)$ $\quad n+\omega = \omega$!

$2\cdot\omega = \ldots = \omega$ $\quad \omega\cdot 2 = \ldots = \omega+\omega$ $\quad +,\cdot$ not commutative

- $\alpha+0 = 0+\alpha = \alpha$ $\quad$ $\alpha\times 0 = 0\times\alpha = 0$ $\quad$ $+,\cdot$ are associative
- $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$ $\quad$ $(\alpha+\beta)\gamma \neq \alpha\gamma+\beta\gamma$ $((1+1)\omega \neq \omega+\omega)$
- $1\cdot\alpha = \alpha\cdot 1 = \alpha$

- thm: If $n, m < \omega$, $n+m = m+n$

Pf: Induction on m. Base case $(m=0)$: $n+0 = 0+n = n$ ✓

Induction step: Sps $n+m = m+n$. WTS: $n+(m+1) = (m+1)+n$. Note,

$n+(m+1) \overset{\text{associat.}}{=} (n+m)+1 \overset{\text{ind. hyp}}{=} (m+n)+1 \overset{\text{associat.}}{=} m+(n+1)$ $[= m+(1+n) = (m+1)+n]$

We need to show $n+1 = 1+n$. Induction on n: Base case $(n=0)$: $1+0 = 0+1 = 1$ ✓

Induction step: Sps $n+1 = 1+n$. WTS: $1+(n+1) = (n+1)+1$. Note,

$1+(n+1) = (1+n)+1 \overset{\text{ind. hyp}}{=} (n+1)+1$ as desired. □

- def: If X is a set of ordinals, $\sup X = \min$ ordinal α s.t. $\beta \leq \alpha$ $\forall \beta\in X$
  - $\sup X = \bigcup X$
- Lemma: let $(X,<)$ be a well ordered set. Sps S is a set of downward closed subsets of X s.t. $X = \bigcup S$. Then $\text{type}(X,<) = \sup_{D\in S} \text{type}(D, <\restriction D)$
- Prop: for $\alpha,\beta\in\underline{Ord}$, $\alpha+\beta = \{\alpha$ if $\beta=0$; $(\alpha+\gamma)+1$ if $\beta=\gamma+1$; $\sup_{\gamma<\beta}(\alpha+\gamma)$ if β is limit
- Prop: for $\alpha,\beta\in\underline{Ord}$: $\alpha\beta = \{0$ if $\beta=0$; $\alpha\gamma+\alpha$ if $\beta=\gamma+1$; $\sup_{\gamma<\beta}(\alpha\gamma)$ if β is limit
  - If $\gamma<\beta$ then $\alpha+\gamma<\alpha+\beta$ and $\alpha\gamma<\alpha\beta$ (strict monotone inc in 2^{nd} argument) but $\gamma+\alpha \leq \beta+\alpha$ and $\gamma\alpha\leq\beta\alpha$
- def: for $\alpha,\beta\in\underline{Ord}$ $\alpha^\beta = \{1$ if $\beta=0$; $\alpha^\gamma\cdot\alpha$ if $\beta=\gamma+1$; $\sup_{\gamma<\beta}\alpha^\gamma$ if β is limit

- def/Application: Given $n<\omega$, $2\le b<\omega$, the base b expansion of n is $n=b^k\cdot d_k+b^{k-1}d_{k-1}+\dots+bd_1+d_0$ where $k<\omega$ and $0\le d_i\le b-1$, called the digits of the expansion. The expansion is unique w/ $d_k\neq 0$.
- def: hereditary base-2 expansion: write each exponent in base 2, each of those exponents in base 2, etc... $100=2^{2^2+2}+2^{2^2+1}+2^2$
- def: a Goodstein sequence is a sequence $(n_b)_{b\ge 2}$ of natural numbers s.t. for all $b\ge 2$ 1) if $n_b=0$, $n_{b+1}=0$ 2) else, write the hereditary base-b expansion of n_b and replace every b by $b+1$, then subtract 1 (this is n_{b+1})
ex: $n_2=100=2^{2^2+2}+2^{2^2+1}+2^2$ $n_3=3^{3^3+3}+3^{3^3+1}+3^3-1\approx 2\cdot 10^{14}$ (big number :))
$=3^{3^3+3}+3^{3^3+1}+3^2\cdot 2+3\cdot 2+2$, $n_4=4^{4^4+4}+4^{4^4+1}+4^2\cdot 2+4\cdot 2+2-1\approx 3\times 10^{156}$
- thm (Goodstein): Every Goodstein sequence reaches 0 in finitely many steps
ex $n_2=3$ takes 7 steps to reach 0, $n_2=4$ takes $\approx 3\cdot 2^{4\cdot 10^8}$ steps in finite steps
Pf idea: replace the base by ω, then the "-1" gives a decreasing seq of ordinals which must reach 0
$F_b(0)=0$, $F_b(n)$ = write n in hered. base b, replace 'b' with 'ω',
Lemma: for each $2\le b<\omega$, $F_b:\omega\to\mathrm{Ord}$ is strictly increasing (pf: by induction)
let $(n_b)_b$ be a G. sequence, let $\alpha_b=F_b(n_b)\in\mathrm{Ord}$. let (if $n_b\neq 0$)
m_b = number from n_b replacing $b\to b+1$. Then $n_{b+1}=m_b-1$, so
$\alpha_b=F_b(n_b)=F_{b+1}(m_b)=F_{b+1}(n_{b+1}+1)\underset{\text{lemma}}{>}F_b(n_{b+1})=\alpha_{b+1}$
but we cannot have an infinite decreasing seq of ordinals. $\square$
- def: for two cardinals κ,λ, let $\kappa\oplus\lambda=|\kappa\sqcup\lambda|$ well orderable so has a cardinality
and $\kappa\otimes\lambda=|\kappa\times\lambda|$ (also well orderable)
- Facts: $n,m<\omega$ then $n+m=n\oplus m<\omega$, $n.m=n\otimes m<\omega$,
$\oplus,\otimes$ are associative and commutative, left & right distributive
- thm: for any infinite cardinal κ, $\kappa\otimes\kappa=\kappa$
$\Rightarrow$ if one of κ,λ is infinite, $\kappa\oplus\lambda=\max(\kappa,\lambda)$, if nonzero, $\kappa\otimes\lambda=\max(\kappa,\lambda)$
- fact: 1) (AC) If A is infinite, $A\approx A\times A$
2) If A,B well-ord. $|A\cup B|=\max\{|A|,|B|\}$
3) B infinite, well-ord., $A\subseteq B$, $|A|<|B|$, then $|B\setminus A|=|B|$

AC: given A, which injection to $\bigcup X$? recording which set a came from

4) (AC) Ctbl union of ctbl sets is ctbl

Pf: X is ctbl $\Rightarrow X \precsim \omega \Rightarrow f: X \to \omega$ injection. def: $\bigsqcup X := \{(A,a) : a \in A \in X\}$.
$\bigsqcup X \to \bigcup X$ surjection by $(A,a) \mapsto a$. each $A \in X$ ctbl, $\exists g_A : A \to \omega$ injective
define $h: \bigsqcup X \to \omega \times \omega$ by $h(A,a) = (f(A), g_A(a))$, injective
So $\bigsqcup X \precsim \omega \times \omega = \omega$, and hence $\bigsqcup X$, $\bigcup X$ are countable. □

5) (AC) If κ is an inf. cardinal, then the union of $\leq \kappa$ sets of card $\leq \kappa$, has card $\leq \kappa$

(AC on): def for cardinals κ, λ, let $\kappa^\lambda = |{}^\lambda \kappa|$ set of all $f: \lambda \to \kappa$

- note: ordinal exp $\neq$ cardinal exp. ω refers to the ordinal, $\aleph_0$ refers to the cardinal
  $\triangleright$ $2^\omega = \sup_{n<\omega} 2^n = \omega$, but $2^{\aleph_0} = |\mathcal{P}(\omega)|$

6) $|\mathcal{P}(\kappa)| = 2^\kappa$ $\forall \kappa \in \underline{\text{Card}}$: $2^\kappa = |{}^\kappa 2| = |\{f: \kappa \to \{0,1\}\}|$. $f \mapsto \{\gamma < \kappa : f(\gamma) = 1\}$

7) If $\lambda \geq \omega$, $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$ $2^\lambda > \lambda$

Pf: $\kappa \geq 2 \Rightarrow 2^\lambda \leq \kappa^\lambda$. For other direction: $2^\lambda = 2^{\lambda \otimes \lambda} \overset{?}{=} (2^\lambda)^\lambda \geq \lambda^\lambda \geq \kappa^\lambda$ □

${}^{\lambda \times \lambda}2$: set of fns $f: \lambda \times \lambda \to 2$ currying

${}^\lambda({}^\lambda 2)$: set of fns $F: \lambda \to ({}^\lambda 2)$

- def: $x \in \mathbb{R}$ is algebraic if it is a root to a poly w/ integer coeffs.
  transcendental otherwise

- thm: $\exists$ transcendental numbers
  Pf: polys of deg $\leq d$ w/ integer coeffs $\approx {}^{d+1}\mathbb{Z} \approx \omega$. So
  |polys w/ intgr coeffs| $\leq \omega \otimes \omega = \omega$, each poly has finite # roots. □

- note: If κ is a cardinal, $\kappa = \{\alpha \in \underline{\text{Ord}} : \alpha < \kappa\} = \{\alpha \in \underline{\text{Ord}} : |\alpha| < \kappa\}$, $\alpha < \kappa \Leftrightarrow |\alpha| < \kappa$ (clear)

- thm (Lindenbaum): If $f: \mathbb{R} \to \mathbb{R}$ is any function, then f is a sum of 2 injective fns

Pf. fix a bijection $2^{\aleph_0} \to \mathbb{R}$ by $\alpha \mapsto x_\alpha$ ($2^{\aleph_0} \approx \mathbb{R}$), so $\mathbb{R} = \{x_\alpha : \alpha < 2^{\aleph_0}\}$
Recursion on $\alpha < 2^{\aleph_0}$: Suppose $\alpha < 2^{\aleph_0}$ and all $g(x_\beta)$, $h(x_\beta)$ have been determined for all $\beta < \alpha$. Define $A_\alpha = \{g(x_\beta) : \beta < \alpha\}$ (want $g(x_\alpha) \notin A_\alpha$ for injectivity of g), and $B_\alpha = \{f(x_\alpha) - h(x_\beta) : \beta < \alpha\}$ (want $h(x) \notin B_\alpha$ for injectivity of h). Then $|A_\alpha|, |B_\alpha| \leq |\alpha|$ so $|A_\alpha \cup B_\alpha| \leq |\alpha| \oplus |\alpha| < 2^{\aleph_0}$
and so $\mathbb{R} \neq A_\alpha \cup B_\alpha$. (AC): Choose $y \in \mathbb{R} \setminus (A_\alpha \cup B_\alpha)$, define $g(x_\alpha) = y$.
and $h(x_\alpha) = f(x_\alpha) - y$ □

↳ true in ZF$^-$

▷ Additive Ramsey Theory

- thm: If $k < \omega$ and $f: \omega \to k$ (color the naturals), then there exist distinct $a, b, c, d \in \omega$ s.t. $f(a) = f(b) = f(c) = f(d)$ and $a + b = c + d$

Pf: Fix $f: \omega \to k$

Step 1: For each $x \in \omega$, there exist $a < b < k+1$ and $i < k$ s.t. $f(a+x) = f(b+x) = i$ (Pigeonhole).

Let $T(x)$ be an arb. triple $(a, b, i) \in (k+1) \times (k+1) \times k$ s.t. above holds

Step 2: There are $x < y < z$ s.t. $T(x) = T(y) = T(z)$ (Pigeonhole, finite $< \omega$)

Step 3: There are $x < y$ s.t. $T(x) = T(y) = (a, b, i)$ and $y - x \neq b - a$

Pf: from step 2, both $y - x$ and $z - x$ cannot be $b - a$. ◻

Now, we have that $f(a+x) = f(b+x) = f(a+y) = f(b+y) = i$

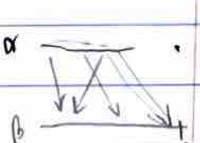
and $(a+x) + (b+y) = (b+x) + (a+y)$ distinct ($b + x \neq a + y$ b/c $y - x \neq b - a$) ◻

- Q: Given $f: \mathbb{R} \to \omega$ (color reals w/ countable colors) does statement still hold?
- thm: "Yes" iff CH fails ($|\mathbb{R}| = 2^{\aleph_0} > \aleph_1$)
- lemma/def: for a set $X \subseteq \mathbb{R}$, define $\hat{X}$ as follows: $X = X_0$, $X_{n+1} = X_n \cup \{a+b, a-b : a, b \in X_n\}$, $\hat{X} = \bigcup_{n \in \omega} X_n$ (additive group generated by X), then 1) $X \subseteq \hat{X}$, 2) $\hat{X}$ is closed under $+, -$ 3) $X \subseteq Y \Rightarrow \hat{X} \subseteq \hat{Y}$ 4) X countable $\Rightarrow \hat{X}$ countable

▷ Cofinality

- $\aleph_\omega = \sup_{n \in \omega} \aleph_n = \bigcup_{n \in \omega} \aleph_n$ - first limit cardinal
- def: Let k be a cardinal. The cofinality $cf(k)$ is the least cardinal λ s.t. k is the union of λ sets of cardinality $< k$
  - $\aleph_0 \leq cf(\aleph_\omega) \leq \aleph_0$

can't have finite union — we wrote it as union of ω sets



- def: Let $\alpha, \beta \in \mathrm{Ord}$. A function $f: \alpha \to \beta$ is cofinal (in β) if $\forall \gamma < \beta, \exists \delta < \alpha$ s.t. $f(\delta) \geq \gamma$

get arb. close to β

- the cofinality of β, $cf(\beta)$ is the least α s.t. $\exists f: \alpha \to \beta$ cofinal

ex: $cf(0) = 0$, $cf(1) = 1$, $cf(2) = 1$ ($\alpha = 1$) $cf(n) = 1$ ($0 < n < \omega$)

$cf(\omega) = \omega$, $cf(\omega+1) = 1$, $cf(5) = 1$ ($\beta = 5$) $cf(\beta+1) = 1$ ($\beta \in \mathrm{Ord}$)

$cf(\alpha) \geq \omega$ $\forall$ limit ord α.

$2\alpha = \alpha \iff \alpha$ limit

$\alpha = \beta+1 \Rightarrow \alpha \leq 2\beta+1 < 2\beta+2 = 2\beta+1 = 2\alpha$

composition of cf. fns. is cf.

- Facts: 1) $cf(\beta)$ is a cardinal ($|cf(\beta)| \approx cf(\beta) \to \beta \Rightarrow cf(\beta) = |cf(\beta)|$)
  2) $cf(\beta) \leq |\beta| \leq \beta$ ($|\beta| \to \beta$ is cofinal fn) ($cf(\aleph_\omega) = \omega$, $n \mapsto \aleph_n$)
- Lemma: let $\beta \in \mathrm{Ord}$. Then $\exists$ strictly increasing cofinal fn $cf(\beta) \to \beta$
- Lemma: let $\alpha, \beta \in \mathrm{Ord}$. If $\exists$ strictly inc cofinal fn $\alpha \to \beta$, then $cf(\alpha) = cf(\beta)$
  ($cf(\alpha) \geq cf(\beta)$): $cf(\alpha) \to \alpha \to \beta$
- Corollaries: 1) $cf(\beta) = cf(cf(\beta))$ 2) If α is a limit ord, $cf(\aleph_\alpha) = cf(\alpha)$
- Thm (AC): If κ is an infinite cardinal, then $cf(\kappa^+) = \kappa^+$ (κ^+ is smallest cardinal w/ strictly greater cardinality)
- ex $cf(\aleph_1) = \aleph_1$, $cf(\aleph_{\omega+5}) = \omega+5$ $cf(\aleph_{\aleph_3}) = cf(\aleph_3) = \aleph_3$
- def: an infinite cardinal κ is regular if $cf(\kappa) = \kappa$, singular otherwise
  ex: every ∞ successor cardinal is regular, $\aleph_\omega$ is regular
- def: a cardinal κ is weakly inaccessible if $\kappa > \omega$, regular, and a limit cardinal
- def: κ is inaccessible if $\kappa > \omega$, regular, and $\kappa > 2^\lambda$ for all $\lambda < \kappa$
  - κ weakly inaccessible $\Rightarrow \kappa = \aleph_\alpha$ for some limit ordinal α ($\alpha \leq \kappa$)
    $\kappa = cf(\kappa) = cf(\aleph_\alpha) = cf(\alpha) \leq \alpha \Rightarrow \alpha = \kappa$ so $\kappa = \aleph_\kappa$ ö w/o AC,
  - $\lambda := \sup(\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots)$ then $\lambda = \aleph_\lambda$ but $cf(\lambda) = \omega$ ⊥
- Lemma (König)(AC): If κ is an infinite cardinal then $\kappa^{cf(\kappa)} > \kappa$ (no surjection $\kappa \to {}^{cf(\kappa)}\kappa$
  - Corollary: $cf(2^\kappa) > \kappa$ pf: If $cf(2^\kappa) \leq \kappa$, then $2^\kappa < (2^\kappa)^{cf(2^\kappa)} \leq (2^\kappa)^\kappa = 2^{\kappa \otimes \kappa} = 2^\kappa$ ↯ □
- Consider class fn $F(\kappa) = 2^\kappa$ (for infinite cardinals κ)
  - König: $cf(F(\kappa)) > \kappa$
  - $\kappa \leq \lambda \Rightarrow F(\kappa) \leq F(\lambda)$
- Easton's Thm: any fn F satisfying the above can be the map $\kappa \mapsto 2^\kappa$ on regular card κ.
  (this is the only restrictions on maps $\kappa \to 2^\kappa$)
- Silver's thm: $\aleph_\lambda$ (or any singular cardinal of unctbl cofinality) can't be smallest counterex for GCH
- PCF theory (Shelah): If $2^{\aleph_n} < \aleph_\omega$ $\forall n < \omega$ then $2^{\aleph_\omega} < \aleph_{\aleph_4}$

AC OFF:
- CH: $\omega \lesssim A \lesssim \mathcal{P}(\omega)$ then $\omega \approx A$ or $A \approx \mathcal{P}(\omega)$ (AC: $2^{\aleph_0} = \aleph_1$)
- GCH: If X infinite and $X \lesssim A \lesssim \mathcal{P}(X)$, then $A \approx X$ or $A \approx \mathcal{P}(X)$ (AC: $2^\kappa = \kappa^+$)
- Thm (Sierpinski): GCH $\Rightarrow$ AC.
  Pf idea: let X infinite. Show: $X \lesssim X \times X \lesssim \mathcal{P}(X)$ and $\mathcal{P}(X) \not\lesssim X \times X$,
  by GCH, $X \approx X \times X$, Tarski's thm $\Rightarrow$ AC □

α lat $\Rightarrow \alpha = \omega \cdot \beta$

$2\alpha = 2\omega \cdot \beta = \omega \cdot \beta = \alpha$

finite seq of X

$\bigcup_{n<\omega} {}^{n}X = {}^{<\omega}X$

no AC ↓

- Thm (Halbeisen-Shelah) (no AC): If X set, $\omega \lesssim X$, then $\mathcal{P}(X) \not\lesssim {}^{<\omega}X$
- Thm (Specker): If $5 \lesssim X$, then $\mathcal{P}(X) \not\lesssim X \times X$
- Lemma: If X infinite, $\mathcal{P}(X) \not\lesssim X \sqcup 1$ (can't inject w/ one extra elt)
- Claim: $X \sqcup 1 \approx X \iff \omega \lesssim X$

Pf of Claim: Just assume X, 1 disjoint for notation, work w/ $X \cup \{0\}$.
($\Leftarrow$): Let $\omega \to X$, $n \mapsto x_n$ injection. $f: X \cup \{0\} \to X$ by
$f(x) = \{ x_0$ if $x = 0$; x_{n+1} if $x = x_n$ some n; x else   ($\Rightarrow$) exercise. ∎

Pf of Lemma: We know $\mathcal{P}(X) \not\lesssim X$. So if $\omega \lesssim X$, then $X \approx X \sqcup 1$ implies $\mathcal{P}(X) \not\lesssim X \sqcup 1$. So assume FSOC that $f: \mathcal{P}(X) \to X \sqcup 1$ injection, (we will show $\omega \lesssim X$, contradiction) WLOG we can assume $f(X) = 0$. now define $x_0 = f(\emptyset)$, $x_1 = f(\{x_0\})$, $x_2 = f(\{x_0, x_1\})$, ...
$n \mapsto x_n$ is an injection $\omega \to X$ bc f is injective and each $\{x_0, ..., x_n\}$ is finite so not $= X$, hence $x_n \in X$ (not $\in \{0\}$) and x_n are distinct since $\{x_0, ..., x_n\}$ are diff sizes and f injective. ∎

- GCH $\Rightarrow$ for every infinite X, $X \sqcup 1 \approx X$ and $\omega \lesssim X$
- Lemma: $\exists$ class fn $\mathrm{Ord} \setminus \omega \to \mathcal{U}: \alpha \mapsto q_\alpha$ st. for every ordinal α, q_α is a bijection $\alpha \times \alpha \to \alpha$.
- thm: $\exists$ class fn $\mathrm{Ord} \setminus \omega \to \mathcal{U}: \alpha \mapsto p_\alpha$ st. for every ordinal α, p_α is a bijection ${}^{<\omega}\alpha \to \alpha$
- Cantor Normal Form: writing ordinals in base ω
- thm: for every $\alpha \in \mathrm{Ord}$, $\exists$ unique, finite seq of ordinals $\beta_1 > \beta_2 > ... > \beta_k$ and >0 integers $d_1, ..., d_k$ st. $\alpha = \omega^{\beta_1} \cdot d_1 + \omega^{\beta_2} \cdot d_2 ... + \omega^{\beta_k} \cdot d_k$
- note: $\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot}}}} = \sup(\omega, \omega^\omega, \omega^{\omega^\omega} ...)$ then $\omega^{\varepsilon_0} = \varepsilon_0$ (CNF has $\alpha = \beta_1$)
- Goal: GCH $\Rightarrow$ AC ($\Leftrightarrow X \approx X \times X$ for inf. X)
  Steps: $\omega \lesssim X \Rightarrow \mathcal{P}(X) \not\lesssim {}^{<\omega}X$ $5 \lesssim X \Rightarrow \mathcal{P}(X) \not\lesssim X \times X$
  Want explicit bijection $\alpha \approx \alpha \times \alpha$, α inf ordinal

Pf: existence: Induction on α. $\alpha = 0$, $0 = 0$ is CNF
Ind: Let $\alpha > 0$, sps all smaller ord have CNF:
$S = \{\beta \in \mathrm{Ord}: \omega^\beta \leq \alpha\} \subseteq \alpha$.
(set)

$\sup S = \max S$

Claim: S has a largest elt. $\omega^{\sup S} = \sup\{\overbrace{\omega^\beta}^{\leq\alpha} : \beta \in S\} \leq \alpha$ so $\sup S \in S$ [...]

def: for any two ordinals α, β, their fusion $\alpha * \beta$ is defined:

CNF: $\gamma_1 > \gamma_2 > \dots > \gamma_k$ and $c_1, \dots, c_k, d_1, \dots, d_k < \omega$ s.t. could be 0

$\alpha = \omega^{\gamma_1} c_1 + \dots + \omega^{\gamma_k} c_k$, $\beta = \omega^{\gamma_1} d_1 + \dots + \omega^{\gamma_k} d_k$ $\quad f(0,0) = 0$

$\alpha * \beta := \omega^{\gamma_1} f(c_1, d_1) + \dots + \omega^{\gamma_k} f(c_k, d_k)$ (for a bijection $f: \omega \times \omega \to \omega$)

↳ $\mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord} : (\alpha, \beta) \mapsto \alpha * \beta$ is a bijection (b/c f is bijection)

$\alpha \times \alpha \approx \omega^\rho \times \omega^\rho \overset{*}{\approx} \omega^\rho \approx \alpha$

- thm: $\exists$ class fn $\mathrm{Ord} \setminus \omega \to V$ st. for each $\omega \leq \alpha$ ord, ρ_α is a bijection ${}^{<\omega}\alpha \to \alpha$
- (Specker) Pf: If $\omega \lesssim X$, then $\mathcal{P}(X) \not\lesssim X \times X \leq {}^{<\omega}X$ (sq'ing of larger, 2)

Strat: Given an injection $f: \mathcal{P}(X) \to X \times X$, we'll construct an injection $\omega \to X$

Fact 1: If X is finite, $|X| = n < \omega$, then $\mathcal{P}(X)$ is finite too, and $|\mathcal{P}(X)| = 2^n$

Moreover, there is a function $\omega \to V : n \mapsto \varphi_n$ st. $\varphi_n : \mathcal{P}(n) \to 2^n$ is a bijection (a canonical choice for φ_n bijection)

Fact 2: If $5 \leq n < \omega$ then $2^n > n^2$

$\Rightarrow$ If X finite, $\mathcal{P}(X) \not\lesssim X \times X$ b/c $|\mathcal{P}(X)| = 2^n$, $|X \times X| = n^2$

So assume X infinite, as $5 \lesssim X$, then $a_0, \dots, a_4 \in X$ distinct exist.

$f: \mathcal{P}(X) \to X \times X$ injective [WANT: injection $\omega \to X : n \mapsto a_n$] recursively

Sps for some $5 \leq n < \omega$, $a_0, \dots, a_{n-1} \in X$ determined.

[WANT to choose $a_n \in X \setminus S$, $S = \{a_i : i < n < n-1\}$] - well order $\mathcal{P}(S) \to 2^n$ □

- $GCH \Rightarrow AC$

Pf: let X infinite. $GCH \Rightarrow \omega \lesssim X$. So $\mathcal{P}(X) \not\lesssim {}^{<\omega}X$. $X \lesssim X \times X \overset{?}{\lesssim} \mathcal{P}(X)$

Claim 0: $X \approx X \sqcup 1$ ($\Leftarrow \omega \lesssim X$)

Claim 1: $X \approx X \sqcup X$ Pf: Obv $X \lesssim X \sqcup X$. $\mathcal{P}(X) \sqcup \mathcal{P}(X) \approx \mathcal{P}(X \sqcup \{0\})$

by $(A, i) \mapsto \{A$ if $i = 0$, $A \cup \{0\}$ if $i = 1$. $X \lesssim X \sqcup X \lesssim \mathcal{P}(X) \sqcup \mathcal{P}(X) \approx \mathcal{P}(X \sqcup 1) \approx \mathcal{P}(X)$

But HS: $\mathcal{P}(X) \not\lesssim X \sqcup X \lesssim {}^{<\omega}X$ so $GCH \Rightarrow X \approx X \sqcup X$ □

Claim 2: $X \approx X \times X$. Pf: $\mathcal{P}(X) \times \mathcal{P}(X) \approx \mathcal{P}(X \sqcup X)$.

$X \lesssim X \times X \lesssim \mathcal{P}(X) \times \mathcal{P}(X) \approx \mathcal{P}(X \sqcup X) \approx \mathcal{P}(X)$ but $\mathcal{P}(X) \not\lesssim X \times X$

by HS, hence $X \approx X \times X$ by GCH. $\Rightarrow AC$. □∪□

- If $\mathcal{U}$ satisfies ($\models$) ZF$^-$ then $V \models$ ZF
- def. $(\mathcal{U}, \in)$. Let $\mathcal{C}$ be a class, φ be a formula. Write $\varphi^{\mathcal{C}}$ for the formula obtained from φ by restricting quantifiers to $\mathcal{C}$: ex: $\exists x \to \exists x \in \mathcal{C}$, $\forall x \to \forall x \in \mathcal{C}$
- def: If φ is fmla w/o free vars, params from $\mathcal{C}$, then φ holds in $\mathcal{C}$ or $\mathcal{C}$ satisfies φ ($\mathcal{C} \models \varphi$) if $\varphi^{\mathcal{C}}$ is true in $\mathcal{U}$.
  - $A = \{\emptyset, \underset{x}{\{\emptyset\}}, \underset{y}{\{\emptyset, \{\{\emptyset\}\}\}}\}$ does not satisfy Ext
- def: A class is transitive if $x \in \mathcal{C} \to x \subseteq \mathcal{C}$
- lemma: If $\mathcal{U} \models$ Ext, $\mathcal{C}$ transitive, then $\mathcal{C} \models$ Ext.
- def: φ is a Δ_0-formula if every quantifier in φ is bounded, i.e., of the form $\forall x \in y$, $\exists x \in y$ where y is a var or param
  - a property is Δ_0 if it can be expressed by a Δ_0 formula!
  - ex. "$x \subseteq y$" is Δ_0: $\forall z \in x\ (z \in y)$, "f function", etc.
- ☆ Lemma: $\mathcal{C}$ is transitive, φ (Δ_0 fmla) w/ no free vars, possibly params from $\mathcal{C}$. Then $\mathcal{C} \models \varphi \iff \mathcal{U} \models \varphi$ → "α is limit, succ", "n is natural #", $x = \omega$
- If $\mathcal{U} \models$ ZF then "α is an ordinal" is Δ_0.
  → "x is well ordered" is generally not Δ_0 (linear order is Δ_0)
- Verifying axioms: Suppose $\mathcal{U} \models$ ZF$^-$, let $\mathcal{C}$ be transitive
  - $\mathcal{C} \models$ Ext (since $\mathcal{U} \models$ Ext)
  - $\mathcal{C} \models$ Empty $\iff \emptyset \in \mathcal{C}$
  - $\mathcal{C} \models$ Pair $\iff \forall a, b \in \mathcal{C}, \{a, b\} \in \mathcal{C}$ ← Δ_0 formulas
  - $\mathcal{C} \models$ Union $\iff \forall x \in \mathcal{C}, \bigcup x \in \mathcal{C}$ "x is powerset of y" is NOT Δ_0
  - $\mathcal{C} \models$ Pow $\iff \forall x \in \mathcal{C}, \{y \in \mathcal{C} : y \subseteq x\} \in \mathcal{C}$ → is a set (in $\mathcal{U}$) by Comp/Pow
  - If $\omega \in \mathcal{C}$ then $\mathcal{C} \models$ Inf ($\Leftarrow$ only true if $\mathcal{C} \models$ Found)
  - $\mathcal{C}$ satisfies Comprehension $\iff$ for all $x \in \mathcal{C}$, every formula $\varphi(z)$ w/ one free var z, params from $\mathcal{C}$, that $\{z \in x : \mathcal{C} \models \varphi(z)\} \in \mathcal{C}$
  - $\mathcal{C}$ satisfies Replacement $\iff$ for every formula $\varphi(x, y)$, free vars x, y, params from $\mathcal{C}$ s.t. φ defines a class fn in $\mathcal{C}$ (i.e. $\forall x \in \mathcal{C}$, there is at most 1 $y \in \mathcal{C}$ st. $\mathcal{C} \models \varphi(x, y)$) and for every $X \in \mathcal{C}$, we have $\{y \in \mathcal{C} : \exists x \in X\ (\mathcal{C} \models \varphi(x, y))\} \in \mathcal{C}$

◦ AC holds in $\mathcal{C}$ $\Leftrightarrow$ exercise! (same process)

◦ Found holds in $\mathcal{C}$ $\Leftrightarrow$ exercise!

- exercise: If $\mathcal{U} \models ZF$, and $\mathcal{C}$ is any class, then $\mathcal{C} \models$ Found
- ex. $\mathcal{C} = \mathrm{Ord}$ (transitive)

does $\mathcal{C} \models$ Pow? Yes! (even though subsets of some ordinals are not ordinals)

$\forall \alpha \in \mathrm{Ord}$, $P^{\mathrm{Ord}}(\alpha) = \{\beta \in \mathrm{Ord} : \beta \subseteq \alpha\} = \alpha + 1 \in \mathrm{Ord}$. $\mathrm{Ord} \not\models$ Pair!

- ex: $\mathrm{Ord}^{\mathcal{C}} := \{x \in \mathcal{C} : \mathcal{C} \models \text{"}x \text{ is ordinal"}\}$ $\rightarrow \Delta_0$ formulas

$\mathrm{Card}^{\mathcal{C}} := \{x \in \mathcal{C} : \mathcal{C} \models \text{"}x \text{ is a cardinal"}\}$

What are $\mathrm{Ord}^{\mathrm{Ord}}$, $\mathrm{Card}^{\mathrm{Ord}}$?

☆ def: an inner model is a transitive class $\mathcal{C}$ s.t. $\mathrm{Ord} \subseteq \mathcal{C}$.

↳ ex: $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$ ($V_0 = \emptyset$, $V_{\beta+1} = P(V_\beta)$, $V_\alpha = \bigcup_{\gamma<\alpha} V_\gamma$ for α limit)

- thm: If $\mathcal{U} \models ZF^-$, then $V \models ZF$.

↳ if ZF^- is consistent, then ZF is consistent

Pf: $X \in V \Leftrightarrow X \subseteq V$ (!) need to check $V \models ZF^-$

now $V \models$ Found? Found (assuming ZF^-) $\Leftrightarrow \forall x \neq \emptyset\ \exists y \in x\ (x \cap y = \emptyset)$

Want: $\forall \emptyset \neq x \in V$, $\exists y \in V\ (x \cap y = \emptyset)$ (use rank)

- thm: If $\mathcal{U} \models ZFC^-$, then $V \models ZFC$

Assume $\mathcal{U} \models ZF$, AC

- reminder: $\kappa \in \mathrm{Card} \setminus \omega$ regular $\Leftrightarrow$ set of card. κ can't be expressed as the union of $<\kappa$ sets of cardinality $<\kappa$
- thm: If κ is inaccessible, then $V_\kappa \models ZFC$ → a set!
- lemma: If κ is inaccessible, then $|V_\kappa| = \kappa$ and $\forall x$, $x \in V_\kappa \Leftrightarrow x \subseteq V_\kappa$ and $|x| < \kappa$.

Pf: $\kappa = \mathrm{Ord} \cap V_\kappa$ (generally, $\alpha = \mathrm{Ord} \cap V_\alpha$) $\Rightarrow |V_\kappa| \geq \kappa$

for ($\leq$): show $\forall \alpha < \kappa$, $\boxed{|V_\alpha| < \kappa}$ If so, then $\underbrace{V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha}_{\text{union of } \kappa \text{ many sets, each of card } <\kappa} \Rightarrow$

$|V_\kappa| \leq \kappa \otimes \kappa = \kappa$

Induction on $\alpha < \kappa$, use κ inaccessible ($\forall \lambda < \kappa$, $2^\lambda < \kappa$)

($\Rightarrow$): let $x \in V_\kappa$. by transitivity, $x \subseteq V_\kappa$. let $\alpha := \mathrm{rank}(x)$, then $x \in V_\alpha \subseteq V_\kappa$

$\alpha < \kappa$, $x \in V_\alpha \Rightarrow x \subseteq V_\alpha$ (transitive) then $|x| \leq |V_\alpha| < \kappa$ $\boxed{|V_\alpha|<\kappa}$ ✓

not cf: $\exists \delta < \beta$ s.t. $\forall \gamma < \alpha$, $f(\gamma) < \delta$

($\Leftarrow$): let $x \subseteq V_\kappa$, $|x| < \kappa$. By regularity, $x \to \kappa : y \mapsto \mathrm{rank}(y)$ can't be cofinal [κ regular: $\mathrm{cf}(\kappa) = \kappa$, can't have smaller domain cf. fn].

so $\alpha := \sup(\mathrm{rank}(y), y \in x) < \kappa$. So $x \subseteq V_\alpha \Rightarrow x \in V_{\alpha+1} \subseteq V_\kappa$ □

- $\mathcal{U}$-formula: a set in $\mathcal{U}$ "representing" a formula
let #s represent $\dot{=}, \dot{\epsilon}, \dot{\wedge}, \dot{\neg}, \dot{\exists}$. $(0, \ldots, 4)$
$\underline{Var} = \{n \in \omega : n \geq 5\}$ – the variables in $\mathcal{U}$-fmla
$_0\mathcal{F} = \{(\dot{=}, x, y), (\dot{\epsilon}, x, y) : x, y \in \underline{Var}\}$: atomic $\mathcal{U}$-flas $(x=y, x\in y)$
$_{n+1}\mathcal{F} = {}_n\mathcal{F} \cup \{(\dot{\wedge}, f, g), (\dot{\neg} f), (\dot{\exists} x, f) : f, g \in {}_n\mathcal{F}, x \in Var\}$
$n \in \omega$. $\mathcal{F} = \bigcup_{n\in\omega} {}_n\mathcal{F}$ – set of $\mathcal{U}$-flas
for a f'la φ, use $\ulcorner \varphi \urcorner$ to denote corresp'n $\mathcal{U}$-fla
$\star$ $\mathcal{U}$-fla, are finite sets, $\mathcal{F} \subseteq V_\omega$. so $\mathcal{F}$ is countable (V_ω is)
- $\mathcal{F}_{\mathcal{C}}$ = class of $\mathcal{U}$-flas w/ params from $\mathcal{C}$
- def: for $f \in \mathcal{F}$, let complexity: $\text{COMP}(f)$ = least $n < \omega$ s.t. $f \in {}_n\mathcal{F}$
  ↳ induction on complexity

AC, W inf: $|\mathcal{F}_W| = |W|$

- $\mathcal{F}^k_W$: set of all $\mathcal{U}$-flas w/ params from W, exactly k free vars
- Lemma: truth is not definable (no truth fn that says if a $f \in \mathcal{F}^0_{\mathcal{U}}$ is true/false $(\{0,1\})$).
- thm: restricting to a set W, truth fn exists
- def: let W set, $A \subseteq W$ is definable if $\exists f \in \mathcal{F}^1_W$ s.t. $A = \{a \in W : W \models f(a)\}$
then "f defines A in W", the definable powerset is $\mathcal{D}(W) := \{A \subseteq W : A \text{ definable}\}$
  $\star$ ex of definable subsets:
  ↳ W itself: $W = \{a \in W : W \models a \dot{=} a\}$
  ↳ $\emptyset = \{a \in W : W \models \dot{\neg}(a \dot{=} a)\}$
  ↳ one elt subsets: $\{b\} = \{a \in W : W \models (b \dot{=} a)\}$
  ↳ finite subsets: $\{b_1, b_2\} = \{a \in W : W \models (a \dot{=} b_1) \vee (a \dot{=} b_2)\}$ ...
- Lemma (AC): If W is infinite, $|\mathcal{D}(W)| = |W|$, so $\mathcal{D}(W) \neq \mathcal{P}(W)$.
pf: $|W| \leq |\mathcal{D}(W)|$ by $x \mapsto \{x\}$ (it's definable).
$|\mathcal{D}(W)| \leq |\mathcal{F}^1_W| \leq |\mathcal{F}(W)|$ by surjective $\mathcal{F}_W \to \mathcal{D}(W)$:
$f \mapsto \{a \in W : W \models f(a)\}$. And $|\mathcal{F}_W| = |W|$. $\square$
- Lemma: If $X \subseteq Y$, $X \in Y$, then $\mathcal{D}(X) \subseteq \mathcal{D}(Y)$ ($X \in Y$ is necessary!)

goal: $\mathcal{U} \models ZF \Rightarrow L \models ZFC + GCH$

(if ZF consistent, ZFC+GCH is consistent)

- $L_\alpha = \begin{cases} \emptyset & \alpha = 0 \\ D(L_\beta) & \alpha = \beta+1 \\ \bigcup\{L_\gamma \mid \gamma < \alpha\} & \alpha \text{ limit} \end{cases}$ then $L = \bigcup\{L_\alpha : \alpha \in \underline{Ord}\}$ is Gödel constructible universe.
- def: set x is constructible if $x \in L$.

$\text{ordr}(x) = \min(\alpha \in \underline{Ord} : x \in L_\alpha)$

- Lemma: $\alpha \in \underline{Ord}$. $\forall \gamma < \alpha$, $L_\gamma \subseteq L_\alpha$ and L_α is transitive, so L is transitive.
- ex: $\alpha \in \underline{Ord}$. $L_\alpha = \bigcup_{\gamma<\alpha} D(L_\gamma)$. $\text{ordr}(x)$ is successor ordinal.

$y \in x \Rightarrow \text{ordr}(y) < \text{ordr}(x)$ (constructible x, y). $L_\alpha \subseteq V_\alpha$ $\forall \alpha$, so $L \subseteq V$.

for finite W, $D(W) = P(W)$, so $L_n = V_n$ $\forall n < \omega$, and also $L_\omega = V_\omega$.

But $L_{\omega+1} \neq V_{\omega+1}$, as $L_{\omega+1}$ is ctbl but $V_{\omega+1} = P(V_\omega)$ isn't.

- Lemma: $\alpha \in \underline{Ord}$ is constructible w/ $\text{ordr}(\alpha) = \alpha+1$. [$\text{Ord} \cap L_\alpha = \alpha$]
- Cor: L is an inner model (transitive and $\underline{Ord} \subseteq L$)
- thm: If $\mathcal{U} \models ZF$ then $L \models ZFC + GCH$. Hence, if ZF is consistent, ZFC+GCH is also.

↳ If every model of A has an inner model satisfying B, then (if A consistent, then B consistent)

- def: a fla w/o params is arithmetical if all quantifiers are bounded by V_ω
- Cor: If an arithmetical stmt is proved in ZFC+GCH, then it can be pvd in ZF.
- Lemma: If $\mathcal{U} \models ZF$, then $\boxed{L \models ZF}$
- def: $\mathcal{C} = \bigcup\{C_\alpha : \alpha \in \underline{Ord}\}$ is a stratified class if $\text{Ord} \to \mathcal{U} : \alpha \mapsto C_\alpha$ is dfn fn st.

1) $\alpha \leq \beta \Rightarrow C_\alpha \subseteq C_\beta$ 2) α limit $\Rightarrow C_\alpha = \bigcup\{C_\gamma : \gamma < \alpha\}$

ex: $V = \bigcup\{V_\alpha : \alpha \in \text{Ord}\}$, $L = \bigcup\{L_\alpha : \alpha \in \text{Ord}\}$

- def: $\mathcal{D} \subseteq \mathcal{C}$ classes, $\varphi(x_1, \dots, x_n)$ formula w/ free vars, no params.

φ is absolute between $\mathcal{D}, \mathcal{C}$ if $\forall a_1, \dots a_n \in \mathcal{D}$, $\mathcal{D} \models \varphi(a_1, \dots a_n) \Leftrightarrow \mathcal{C} \models \varphi(a_1, \dots a_n)$

- thm: (Reflection): let $\mathcal{C} = \bigcup\{C_\alpha : \alpha \in \text{Ord}\}$ be a stratified class, let $\varphi_1, \dots \varphi_n$ be a list of formulas w/o params. Then $\exists \beta \in \underline{Ord}$ st all φ_i are absolute btw $C_\beta, \mathcal{C}$.
- Cor: $\forall \gamma \in \underline{Ord}$, $\exists \beta \geq \gamma$ st $\varphi_1, \dots \varphi_n$ are absolute btw C_β and $\mathcal{C}$.

→ anything true in stratified class will hold in some initial segment

- Axiom of Constructibility: $\mathcal{U} = L$ (every set is constructible)
- thm: $L \models$ Axiom of Constructibility $\exists$ $\forall y \in \dots$
- def: Σ_1 formula if form of (existence & bounded universal quantifiers) + Δ_0-formula

  Π_1 formula if form of (universal & bounded existential quantifiers) + Δ_0-formula

negation of Σ_1 is Π_1 $\forall$ $\exists x \in \dots$

conjunction of Π_1 f'las is equiv to some Π_1

- Σ_1 - class: class defined by Σ_1 - fmla w/o params
  Σ_1 - class fn: same (no params, 2 free vars)
- Lemma: let φ be a Σ_1 - class fn st. dom(φ) is Π_1.
  Then φ is Π_1 as well.
  PF: $\varphi(x)=y$ is Σ_1 fmla (w/o params) by assumption
  $$\varphi(x)=y \Leftrightarrow \underbrace{x\in dom(\varphi)}_{\Pi_1} \wedge \underbrace{\forall y'}_{\Pi_1}(y'=y \vee \underbrace{\neg\varphi(x)=y'}_{\text{negation of }\Sigma_1 \text{ is } \Pi_1}) \quad \square$$
- def: a set A is Σ_1-identifiable if $\{A\}$ is Σ_1
  ↳ there is a Σ_1 fla $\varphi(x)$, no params, 1 f.v. st. $\varphi(x) \Leftrightarrow x = A$
- ex: ω is Σ_1-identifiable ("$x=\omega$" is Δ_0, so it's Σ_1)
- If A is Σ_1-identifiable, quantifiers ranging over A can be written in Σ_1 flas
  $$\forall x\in A(\ldots) \Leftrightarrow \exists y(\underbrace{y=A}_{\Sigma_1} \wedge \forall x\in y(\ldots))$$
- φ is Σ_1 class fn [$y=\varphi(x)$ is Σ_1] then $z\in\varphi(x)$ is
  equiv to Σ_1 fla [$\exists y(y=\varphi(x) \wedge z\in y$]
- thm (Σ_1-recursion): let $\mathcal{E}$ be Σ_1 class fn st. every $\mathcal{E}$-inductive
  function f is in dom($\mathcal{E}$). Then the unique $\mathcal{E}$-ind class fn
  $F: \mathrm{Ord} \to \mathcal{U}$ is Σ_1
- $x \mapsto cl(x)$ is Π_1 (write it out, including minimal transitive set)
  it's also Σ_1 (recursion)
- Lemma: $\mathrm{Ord} \to \mathcal{U}: \alpha \to L_\alpha$ is Σ_1
  - F is Σ_1, but it's also Σ_1 identifiable
- thm: L satisfies Constructibility Axiom ($L \models$ "every set is constructible", diff mean in L)
  - Σ_1 true in small class $\Rightarrow$ holds in big class
- α • exam: $\mathcal{C}$ is an inner model st. $\mathcal{C} \models ZF$, then $L \subseteq \mathcal{C}$ (L is "smallest" FZ)

β (weak) Condensation Lemma: let S transitive set $S \models ZF$ + "all sets constructive", then $S = L_\tau$ for some

γ [next: want $L \models AC$ (actually, $L \models GC$)]
  big idea: U-f'las are well orderable, L is sets defined by U-flas

(ZF)? Lemma: there is a class fn that given a set W, well ordering < on W,
  outputs a well ordering $<^*$ on $\mathcal{D}(W)$

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- GC $\Leftrightarrow$ bijection $\mathrm{Ord} \to U$ $\Leftrightarrow$ $\exists$ well ordering on U
- thm: If $U = L$, then GC holds
  ↳ we can take $U = L$ b/c $U \models ZF \Rightarrow L \models ZF + $ A. of Constructibility
- thm: If $U = L$ then GCH holds $[L \models GCH]$
  ↳ GCH: $2^\kappa = \kappa^+$ $\forall$ infinite cardinals κ [AC holds by prev thm]
- def: a, b sets. b is Σ_1-identifiable over a if $\exists$ Σ_1 f'la $\varphi(x, y)$
  2 free vars, no params st. $x = b \Leftrightarrow \varphi(x, a)$ i.e. $\{b\} = \{x : \varphi(x, a)\}$
  Q: how large can $|b|$ be if b is Σ_1-id. over a?
  $x = \omega$ is Δ_0 so ω is Σ_1-id. over any set (so at least countably inf)
  $x = a$ is Δ_0 (so at least cardinality of a)
  $a = \{b\}$ is Σ_1-identifiable: $x = b \Leftrightarrow x \in a$
  $\Rightarrow tcl(a)$ is Σ_1-idfble over a
- Gödel's Magic Lemma (ZFC): If b is Σ_1-idfble over a then
  $|b| \le \max\{\aleph_0, |tcl(a)|\}$
- note: $|L_\alpha| = |\alpha|$ $\forall \alpha \in \mathrm{Ord}$: $\alpha \subseteq L_\alpha \Rightarrow |\alpha| \le |L_\alpha|$.
- done: $\forall a \in L$, $|ordo(a)| \le \max(\aleph_0, |tcl(a)|)$?
- ex: If $V = L$, then $\forall \alpha \in \mathrm{Ord}$ $V_{\omega+\alpha} \subseteq L_{\aleph_\alpha}$
- ex: $V = L$: $\kappa = \aleph_\kappa \Rightarrow V_\kappa = L_\kappa$
- Löwenheim-Skolem Thm (ZFC): let $A \subseteq B$ sets. Then $\exists A^*$ st. $A \subseteq A^* \subseteq B$,
  $|A^*| \le \max(\aleph_0, |A|)$ and $\forall f \in F^\circ_{A^*}$, $A^* \models f \Leftrightarrow B \models f$
  ▷ A^* looks indistinguishable to B by $\in$-sentences but $|A|$ is bounded.
  ▷ let κ be inaccessible cardinal. then $V_\kappa \models ZFC$. L-S w/ $A = \emptyset$, $B = V_\kappa$
  $\Rightarrow A^* \subseteq V_\kappa$, $|A^*| \le \aleph_0$ (A^* ctbl), for all $f \in F^\circ_{A^*}$, $A^* \models f \Leftrightarrow V_\kappa \models f$
  there is a countable model of ZFC
  - $A^* \models$ "$\exists$ unctbl sets" but this is just saying A^* thinks the set is unctbl
    i.e. $\exists s \in A^*$ st. $A^* \models \neg \exists f (f$ injection $s \to \omega)$ (f just doesn't exist in A^*)
  - $A^* \models \exists s (s = \omega)$, Δ_0, no params, take s that satisfies
    $A^* \models s = \omega$, s as param $\Leftrightarrow V_\kappa \models s = \omega \Rightarrow s = \omega$, so $\omega \in A^*$
  - $\Rightarrow P(\omega) \in A^*$, $V_\omega, V_{\omega+\omega}, V_{\aleph_1}, \ldots \in A^*$ (A^* is not transitive b/c
    uncountable ↝ $V_{\aleph_1} \not\subseteq A^*$)

- Thm: let $\mathcal{C}$ be a class, $\mathcal{C} \models Ext$. Then $\exists$ unique class fn $j: \mathcal{C} \to \mathcal{U}$ st. (MI): j is injective (M2): $ran(j)$ is transitive (M3): $\forall x, y \in \mathcal{C}$, $(y \in x \Leftrightarrow j(y) \in j(x))$. j is Mostowski collapse of $\mathcal{C}$

Pf: elts of $j(x)$ are in range of j by transitivity.

$j(x) = \{ j(y) : y \in \mathcal{C}, y \in x \}$, recursive def on $rank(x)$ $\square$

$\mathcal{C} \models Ext$ used to show injectivity (M1)

- ex: $A = \{\phi, \{\phi\}, \{\phi, \{\{\phi\}\}\}\}$

$\{\phi\}$, $\{\phi, \{\{\phi\}\}\}$ $\not\models Ext$, both contain ϕ but not equal

$j(\phi) = \phi$, $j(\{\phi\}) = \{j(\phi)\} = \{\phi\}$, $j(\{\phi, \{\{\phi\}\}\}) = \{j(\phi)\} = \{\phi\}$ not injective

- ex: $E = \{n \in \omega : n \text{ even}\}$, $E \models Ext$. Mostowski collapse: $j(n) = n/2$
- ex: $\mathcal{C} \subseteq Ord$, proper class. Most. collapse of $\mathcal{C}$ is the (unique) order preserving bijection $\mathcal{C} \to Ord$
- ex: If $\mathcal{C}$ transitive, M.C. is identity.
- ex: If $\mathcal{C} \models Ext$, $A \subseteq \mathcal{C}$ transitive subclass, then j M.C., $j \restriction A = id_A$ (map A ...)
- $j: \mathcal{C} \to \mathcal{C}'$ M. collapse, $\mathcal{C}' = ran(j)$, then j is isomorphism $(\mathcal{C}, \in) \to (\mathcal{C}', \in)$

If φ is a sentence w/ params $a_0, \dots a_n \in \mathcal{C}$, then

$$\mathcal{C} \models \varphi(a_0, \dots a_n) \Leftrightarrow \mathcal{C}' \models \varphi(j(a_0) \dots j(a_n))$$

If $\mathcal{C}$ is a set holds ↗ for all $\mathcal{U}$-fmlas

- Cor: $A \subseteq B$ sets, A transitive; $B \models Ext$. Then $\exists$ transitive set $A' \supseteq A$, st $|A'| \leq \max\{\aleph_0, |A|\}$, and $\forall f \in \mathcal{F}^{o}_{A}$ $A' \models f \Leftrightarrow B \models f$

(not claiming $A' \subseteq B$, so only use params from A)

Pf: L-S: A^*, $A \subseteq A^* \subseteq B$, $|A^*| \leq \max(\aleph_0, |A|)$, $\forall f \in \mathcal{F}^{o}_{A}$, $A^* \models f \Leftrightarrow B \models f$, $B \models Ext \Rightarrow A^* \models Ext$. Tarsky M. collapse, $j: A^* \to \mathcal{U}$, take $A' = ran(j)$. A' transitive ✓, $A' \supseteq A$ b/c A transitive so j is identity on A ✓, if $f \in \mathcal{F}^{o}_{A}$, $A' \models f \Leftrightarrow A^* \models f \Leftrightarrow B \models f$. $\square$

(j is id on A)

$\Rightarrow$ If $\exists$ inaccessible κ card, then $\exists$ transitive countable set satisfying ZFC orange ofange interval sanitives

peanut anton $\aleph_0$ $\aleph$ $\aleph_0$ CH

Descriptive Set Theory

- thm (Cantor): If $C \subseteq \mathbb{R}^n$ is closed, then $|C| \le \aleph_0$ or $|C| = 2^{\aleph_0}$
- def: $P \subseteq \mathbb{R}^n$ is perfect if it's closed and has no isolated pts
- thm (Cantor): If P is nonempty, perfect, then $|P| = 2^{\aleph_0}$
- thm (Cantor-Bendixson): $C \subseteq \mathbb{R}^n$ closed $\Rightarrow$ $C = U \cup P$, $U \cap P = \emptyset$, U ctbl, P perfect

  pf: C closed, $C = U \cup P$, if $P = \emptyset$, then $C = U$ is ctbl. If $P \neq \emptyset$, $|C| \geq |P| = 2^{\aleph_0}$. □
- $A \subseteq \mathbb{R}^n$ has "perfect set property" (PSP) if A is ctbl or A has a nonempty perf. subset

  ↳ closed sets have PSP
- $\exists A \subseteq \mathbb{R}$ st. A, A^c both don't have PSP

"Everything" (7 words)

E	L	F	R	O	C
T	C	E	F	E	S
Y	I	A	T	I	L
O	R	L	N	T	A
N	O	N	H	N	I
Y	A	E	K	D	C
T	R	E	R	H	O
P	M	O	E	C	I