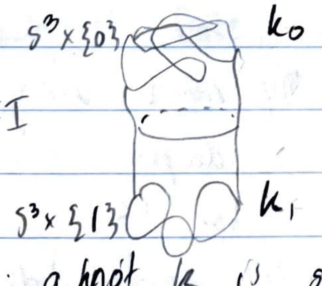
knot Concordance

- fact: for two knots in S^3, they are equivalent ($\exists$ an orientation preserving homeomorphism $\varphi: S^3 \to S^3$) iff they are ambiently isotopic
  - in a general 3-mfd equivalence is weaker than ambient isotopy if MCG is nontrivial
- ({knots in S^3}, #) (connect sum), commutative
  - id = unknot [commutative monoid]
  - inverses? no: if $K \neq$ unknot, then $\nexists J$ st. $J \# K =$ unknot.
- def: a Seifert surface for a knot K is a compact, oriented, connected surface F with $\partial F = K$
  - compact, oriented, connected surfaces classified by genus & # boundary components (Seifert surface for knots have one boundary comp)
- def: $g(K) = \min\{$ genus of $F \mid F$ is Seifert surface for K
  - $g(K) = 0 \Leftrightarrow K$ unknot
  - $g(J \# K) = g(J) + g(K)$ (so no inverses)

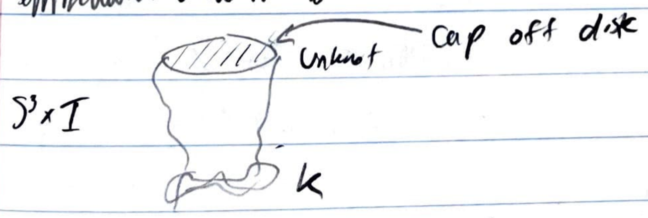
$\chi = 2 - 2g - b$

disks − bands + boundary = 2−2g

★ - def: knots K_0, K_1 in S^3 are smoothly concordant, denoted $K_1 \sim K_2$ if they cobound a smoothly embedded annulus in $S^3 \times I$

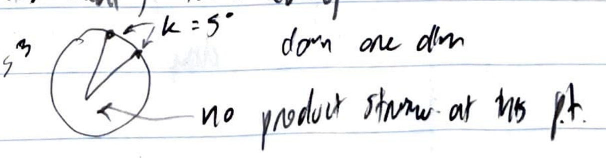


  - topologically concordant ... topologically locally flat (every pt has a product nbhd) embedded annulus
- def: a knot K is smooth / top slice if $K \sim$ unknot
- note: K smoothly (top) slice $\Leftrightarrow K$ bounds a smoothly (top locally flat) embedded disk in B^4



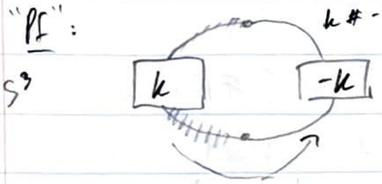
- note: removing all adjectives (only disk embedded) then every knot is slice

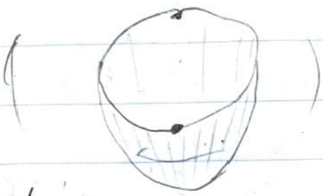
$\mathrm{Cone}(S^3, K) = (B^4, D^2)$



- **def**: mirror of knot: change over crossings ⇔ undercrossings (mK)
- **def**: reverse of knot: change direction of arrow orientation (K^r)
- **def**: minus of knot: do both ($-K = mK^r$)
- **Prop**: $K \# -K$ is smoothly slice
  ↳ want inverses in knot concordance

"Pf":

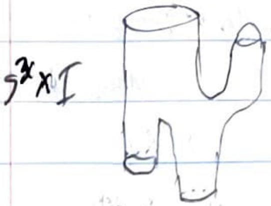




swing this arc in B^4 over to get a sm. embedded disk

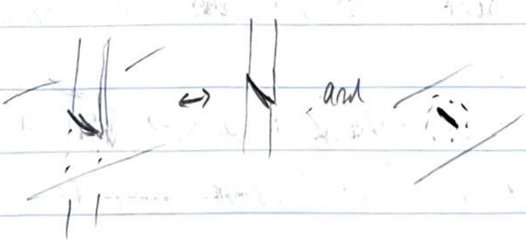
□

- **def**: a knot in S^3 K is ribbon if it bounds a sm. embedded D^2 in B^4 w/ no local max wrt. radial Morse fn. on $f: B^4 \to \mathbb{R}$
  ↳ concordance to unknot w/ no local max



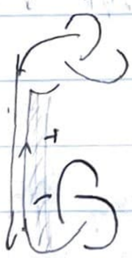
← local max, not ribbon

- **ex**: K ribbon ⇔ K bounds an immersed disk w/ only ribbon singularities



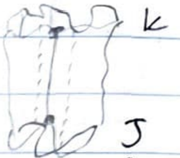
push this deeper
↙ into B^4 for (⇐)
(you get sm embedded disk in B^4)

- $K \# -K$ is ribbon
  - ribbon ⇒ slice ✓
  - slice ⇒ ribbon ??

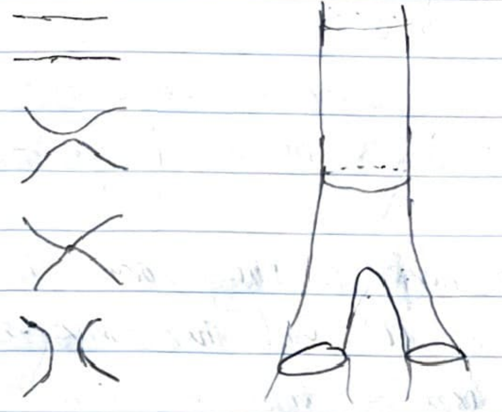


follow this around knot to get a ribbon disk

- exercise: $K \sim J \iff K \# -J$ slice



- exercise: $P(p, -p, n)$ is ribbon ← p odd

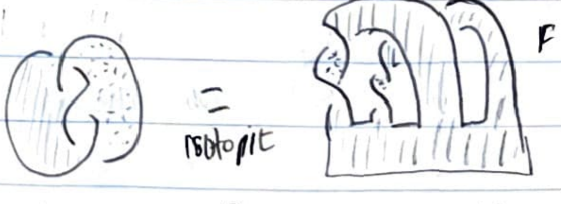


concordance

- def: knot concordance group $\mathcal{C} ::= (\{\text{knots in } S^3\}/\underset{\text{conc}}{\sim}, \#)$
  $\text{id} = [\text{unknot}] = \{\text{slice knots}\}$
  $-[K] = [-K]$

$4_1 \# 4_1 = 4_1 \# -4_1 = [\text{unknot}]$

- which elts finite order? knots isotopic to their own mirror...
  4_1 is isotopic to -4_1, so 4_1 is order at most 2 in $\mathcal{C}$.
  If we can show 4_1 is not slice, it has order 2.



disk-band form, ⇒ basis for $H_1(F)$
(around each band)

- def: the Seifert form for a knot K w.r.t. a Seifert surface F (for K) is the bilinear form $V_F : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$, $(x,y) \mapsto lk(x, y^+)$, y^+ is the positive pushoff of y

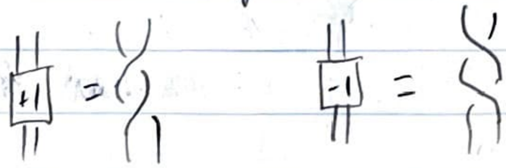
Choose basis for H_1
(go around each band)

Seifert matrix: $$\begin{array}{c|ccc|} & x^+ & y^+ & \cdots \\ \hline x & & & \\ y & & & \\ \vdots & & & \end{array} = V$$

- def: stabilization adds two bands,
  - one of which is untwisted and unknotted
  - other can be twisted, knotted, linked w/ other bands

  on Seifert matrix: $V \to \left[\begin{array}{ccc|cc} & & & \ast & 0 \\ & V & & \vdots & \vdots \\ & & & \ast & 0 \\ \hline \ast & \cdots & \ast & \ast & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right]$ (1)
- def: handle slide: slide end of one band along an adjacent band
  on Seifert matrix: $V \to MVM^T$, M is invertible integer matrix (2)
  (this is a change of basis)
- def: two integer matrices are S-equivalent if they are related by a sequence of operations (1) and (2) and their inverses
- thm: any two Seifert surfaces for a knot K have a common stabilization (might have to stabilize a lot)
- Cor: Any two Seifert matrices for a knot are S-equivalent
  ↳ need an invariant of Seifert matrix that only depends on S-equiv. class to get a knot invariant.
- e.g: Alexander polynomial: $\Delta_K(t) \doteq \det(V - tV^T)$ (up to a mult. of $\pm t^n$)
- e.g: knot signature: $\sigma(K) = \mathrm{sgn}(V + V^T)$ (# of pos evals - # of neg. evals)

in diag/band :



(full twists)

- exercise: show V_1, V_2 Seifert matrices for K_1, K_2, then $V_1 \oplus V_2 = \left[\begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array}\right]$ is a Seifert matrix for $K_1 \# K_2$
  ↳ $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$
  ↳ $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$
- Exercise: If V is a Seifert matrix for K then $-V$ is a Seifert matrix for $-K$

- Prop: If K is top. slice and F is any Seifert surface for K, then $\exists$ basis for $H_1(F;\mathbb{Z})$ st. $V = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, A, B, C $g \times g$ integral square matrices (i.e. V is metabolic: vanishes on half-dim subspace)

Pf sketch: K top slice, w/ slice disk D in B^4, Seifert surface F

then $F \cup D$ bounds a 3-mfd M

Lemma: M compt, connected, oriented 3-mfd st. ∂M is a connected surface of genus g, then $\exists$ basis for $H_1(\partial M, \mathbb{Q})$ st "half lives / half dies" under $i_* : H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q})$

basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ st. $x_i \in \ker(i_*)$, $y_i \notin \ker(i_*)$

$H_1(F) \cong H_1(F \cup D) = H_1(\partial M)$

alt def of lk: $J, K \subset S^3 = \partial B^4$, $lk(J,K) =$ signed intersection # of A, B where A, B are 2-chains in B^4 st. $\partial A = J$, $\partial B = K$. $\square$

- Exercise: $\Delta_K(1) = \pm 1$
- Cor (to Prop above): K slice $\Rightarrow \Delta_K(t) = p(t) \cdot p(t^{-1})$ for some $p(t) \in \mathbb{Z}[t]$. (Fox-Milnor condition)
- def: the determinant of a knot is $|\Delta_K(-1)|$
  $\hookrightarrow \Delta_K(1) = \pm 1$ implies $\det K$ is odd
- Cor: $\det K$ is a square ($\Delta_K(-1) = p(-1) \cdot p(-1)$)
- e.g. Figure-8: $\Delta_K(t) = t - 3 - t^{-1}$, $\det K = 5$ is not square so 4_1 is not slice.
  $\Rightarrow 4_1$ is order two in $\mathcal{C}$
- Cor: K slice $\Rightarrow \sigma(K) = 0$
- Exercise: $\sigma(K)$ is even
- Cor: $\sigma : (\mathcal{C}, \#) \to (2\mathbb{Z}, +)$ is a surj. homomorphism

Pf: (homo): $\sigma(\text{slice}) = 0$, $\sigma(J \# K) = \sigma(J) + \sigma(K)$

(surj): $\sigma(n \cdot \text{RH trefoil}) = -2n$ $\square$

- ex: $0 = \sigma(4_1 \# 4_1) = \sigma(4_1) + \sigma(4_1) \Rightarrow \sigma(4_1) = 0$

Finite order elts have signature zero

- Cor: $\mathcal{C}$ contains a $\mathbb{Z}$-summand: $\mathcal{C} \cong \mathbb{Z} \oplus G$ for some abelian G.

Pf:
$$\begin{tikzcd} 0 \arrow[r] & \ker(\sigma) \arrow[r, hook] & \mathcal{C} \arrow[r, "\sigma/2"] & \mathbb{Z} \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd}$$

$\mathbb{Z}$ free ⇒ ∃ section s s.t. $\frac{\sigma}{2} \circ s = id_{\mathbb{Z}}$. So s.e.s splits as a direct sum $\mathcal{C} \cong s(\mathbb{Z}) \oplus \ker \sigma$. □

- note: RHT is infinite order in $\mathcal{C}$.
- def: Levine-Tristram signatures

  Recall: Hermitian matrix ($A = \bar{A}^T$) is diagonalizable

  Let V be a Seifert matrix, $\omega \in \mathbb{C}$, $|\omega| = 1$.

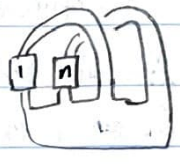
  $V_\omega := (1-\omega)V + (1-\bar{\omega})V^T$

  Fact: if ω is not a root of $\Delta_K(t)$ then V_ω is nonsingular (invertible)

  def: $\sigma_\omega(K) := sgn(V_\omega)$

  ↳ additive on # and vanishes for slice knots

  ↳ this gives a surj homo $\mathcal{C} \to \mathbb{Z}^\infty$ consider these:



- Cor: $\mathcal{C}$ is infinitely generated

  ↳ Countable # of knots in S^3 (enumerate by crossing #) so countably inf. generated
- def: Arf invariant: define a $\mathbb{Z}_2$-quadratic form on $(\mathbb{Z}_2)^{2g}$ by $q(x) = xVx^T$ (V Seifert matrix for K). $Arf(q) := 0$ if q takes on value 0 on majority of elts in $(\mathbb{Z}_2)^{2g}$, 1 if... value 1 on majority...

  $Arf: \mathcal{C} \to \mathbb{Z}_2$
- Fact: $Arf(K) = 0 \Leftrightarrow \Delta_K(-1) = \pm 1 \mod 8$
- def: A symmetric poly $p(t)$ is one such that $p(t^{-1}) = \pm t^n p(t)$
- Recall: K slice ⇒ $\Delta_K(t) = p(t)p(t^{-1})$ for some $p \in \mathbb{Z}[t]$.

  for a poly f irred, symmetric (over $\mathbb{Z}$), biggest expon. of f mod 2 in irred. factorization of $\Delta_K(t)$ is a surj homo $\mathcal{C} \to \mathbb{Z}_2$.

  ⇒ $\mathcal{C}$ has a $(\mathbb{Z}_2)^\infty$ direct summand

  calculate Δ_K for these knots:

- 

basis x, y for $H_1(F)$ (F Seifert surface)

intersection form for $H_1(F)$: $\begin{array}{c|cc} & x & y \\ \hline x & 0 & -1 \\ y & 1 & 0 \end{array}$

- exercise: $V - V^T$ = intersection form for $H_1(F)$
- def: an <u>abstract Seifert form</u> is a bilinear form $V: \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}$ st. $V - V^T$ is unimodular
- exercise: every abstract Seifert form can be realized as a Seifert surface for some knot
- $K \rightsquigarrow V$ (Seifert form)

  $-K \rightsquigarrow -V$

  $K_1 \# K_2 \rightsquigarrow V_1 \oplus V_2$

  K slice $\Rightarrow$ V metabolic

  ↖ $K_1 \sim K_2 \Leftrightarrow K_1 \# -K_2$ is slice
- def: $\mathcal{G} := (\{\text{abstract Seifert forms}\}/\sim, \oplus)$ is the <u>algebraic concordance group</u>

  $V_0 \sim V_1 \Leftrightarrow V_0 \oplus -V_1$ is metabolic

  $\mathcal{C} \to \mathcal{G}$ is a well def. surjective homomorphism

  $[K] \mapsto [V]$

  ↳ Levine 1969: $\mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$

  ↳ Casson - Gordon 1975: $\ker(\mathcal{C} \to \mathcal{G})$ is nontrivial

  ↳ Open: is there n-torsion in $\mathcal{C}$ for $n \neq 2$?
- def: <u>Satellite knots</u>

K (companion)

$S^1 \times D^2$

tubular nbhd of K

↓

$h: S^1 \times D^2 \to \nu(K)$

λ longitude: $S^1 \times \{x\} \mapsto$ 0-framed longitude of K, $x \in \partial D^2$

(((links K 0 times)

(((boundary of a Seifert surface)

writhe(K) = +3

take any knot in solid torus

this knot is Whitehead double (Wh)

<u>exercise</u>: Seifert form for Wh(K) is $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

<u>Cor</u>: $\Delta_{Wh(K)}(t) = 1$

⇒

$P(K) := h(P)$

other patterns:

$(2,1)$ - cable

(or (p,q) -cable $= K_{p,q}$
for coprime p,q)

$w(P) = p$

Mazur pattern

$w(P) = 1$

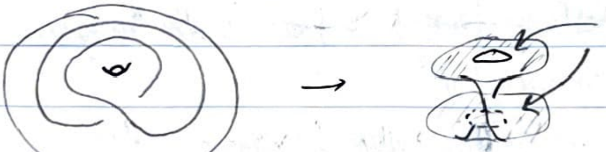
winding number of pattern P

$w(P) := lk(P, \underset{\text{meridian}}{\underset{\uparrow}{\mu}})$

equiv, $w(P) = [P] \in H_1(S^1 \times D^2; \mathbb{Z})$

- If $w(P) = 0$ then P bounds a Seifert surface F in $S^1 \times D^2$, and $h(F)$ is a Seifert surface for $P(K)$.
- If $w(P) \neq 0$ then P is not null homologous in $S^1 \times D^2$ and hence does not bound a surface in $S^1 \times D^2$
  - ↳ but $P \cup \underline{w(P) \text{ longitudes}}$ does bound a surface in $S^1 \times D^2$

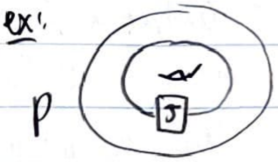
e.g.



build a Seifert surface for $P(K)$ via $S \cup w(P)$ copies of F, where F Seifert surface for K

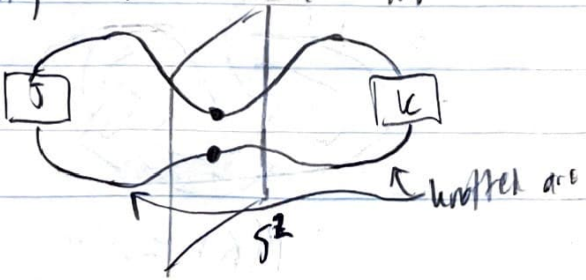
- __thm__: if S, F are minimal genus such surfaces then the resulting surface is a minimal genus Seifert surface for $P(K)$ (K nontrivial)
- __exercise__: $\Delta_{P(K)}(t) = \Delta_K(t^w)\,\Delta_{P(O)}(t)$

  $w = w(P)$, $O =$ unknot
- __ex__:



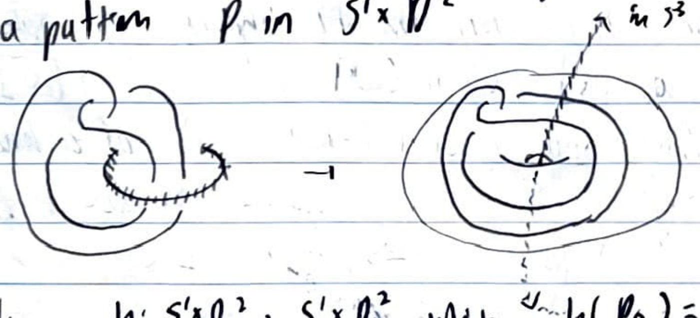
then $P(K) = J \# K$

- key feature of __composite knots__ (can be written as nontrivial connect sum): $\exists$ 2-sphere that intersects K in exactly 2 pts st on either side, the arc is knotted
(not isotopic to an arc in 2 sphere)



- key feature of satellite knots: ∃ essential torus (non boundary parallel, incompressible) in $S^3 - \nu(K)$ (coming from ∂ of the solid torus containing P)
  ↳ for connect sum, swallow - follow

- note: complement of unknot is solid torus
  ⇒ any 2-component link w/ a distinguished unknotted component describes a pattern P in $S^1 \times D^2$



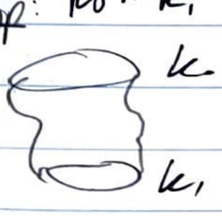
- note: ∃ homo $h: S^1 \times D^2 \to S^1 \times D^2$ which $h(P_0) = P_1$
  so these knots are equiv in $S^1 \times D^2$ but not ambiently isotopic

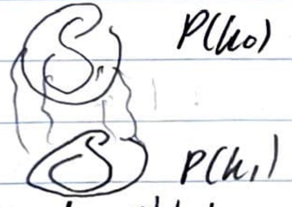
P_0

P_1

* preferred longitude (identification of solid torus w/ $S^1 \times D^2$) matters

- Prop: $K_0 \sim K_1 \Rightarrow P(K_0) \sim P(K_1)$





"satellite the concordance"

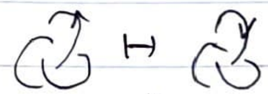
$S_1 \times D_2$ - take a thickened nbhd
in the thickened S^1, consider pattern
so $P: \mathcal{C} \to \mathcal{C}$ is well defined $[K] \mapsto [P(K)]$

eg: $P_0 =$

$P: \mathcal{C} \to \mathcal{C} \equiv id$

or: P_0'

$P: \mathcal{C} \to \mathcal{C}$
$[K] \mapsto [K^r]$



reversible

(are actually isotopic, but not true for all knots)

patterns P_1 in $S^1 \times D^2$ which are concordant (in $S^1 \times D^2 \times I$) to  $= P_0$
induce the same map on concordance ($P_0(K) \sim P_1(K)$)

e.g: P_2:  $P_2 : \mathcal{C} \to \mathcal{C}$: 0 map

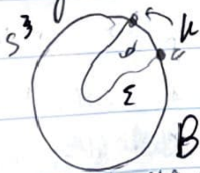
e.g:  (2,1) - cable, $P(K) = K_{2,1}$

Heegaard-Floer homology: sm. concordance invariant $\tau : \mathcal{C} \to \mathbb{Z}$
$\tau(RHT_{2,1}) = 2$ $\tau(LHT_{2,1}) = -1$
So $P_{2,1} : \mathcal{C} \to \mathcal{C}$ is not a homomorphism (2 and -1 are not inverses)

- Conjecture: the only homomorphisms of $\mathcal{C}$ induced by satellite operations are $[K] \mapsto [K]$, $[K] \mapsto [K^r]$ and $[K] \mapsto [0]$

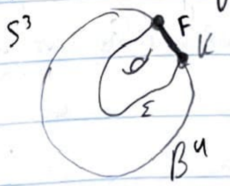
g_4: sm
g_4^{top}: top

- def: the smooth/top slice genus (4-ball genus) of a knot $K \subset S^3$ is $g_4(K) := \min\{ g(\Sigma) : \Sigma$ a sm/top locally flat surface in B^4 w/ $\partial\Sigma = K\}$



  - K slice $\iff g_4(K) = 0$
  - $g_4^{top}(K) \le g_4^{smooth}(K) \le g(K)$
  - $g_4(RHT) = g_4(LHT) = 1$ (LHT/RHT not slice)

- exercise: $|\sigma(K)/2| \le g_4^{top}(K)$

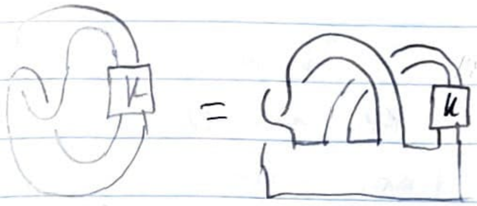
Hint: 

closed surface $F \cup \Sigma$
don't have half lives/half dies but something close

  - $g_4^{top}(RHT \# RHT) = 2$ (σ is additive)

- note: $g_4(K_1 \# K_2) \neq g_4(K_1) + g_4(K_2)$
  - $g_4(LHT \# RHT) = 0$ (LHT # RHT is slice)

- we can build a slice surface for $P(K)$ in a similar way to building a Seifert surface for $P(K)$, but this is generally not minimal.
  - e.g. $K \# -K$ (# as a satellite): $g(K \# -K) = 2g(K)$, but $g_4(K \# -K) = 0$ slice

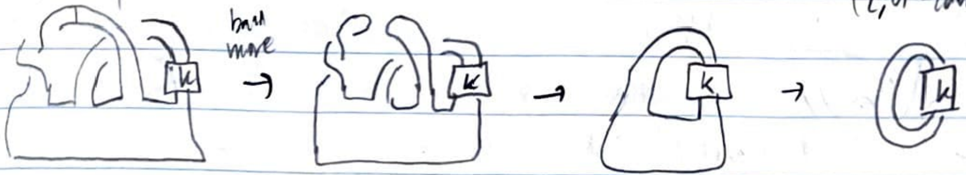


Wh(K)

Seifert form
$$V = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \Delta_{Wh(K)}(t) = 1$$

- thm (Freedman)(hard): If $\Delta_J(t) = 1$ then J is topologically slice
  ↳ so Wh(K) is top. slice ∀K
- recall: K slice → Wh(K) slice ($K \sim U \Rightarrow Wh(K) \sim Wh(U) = U$)

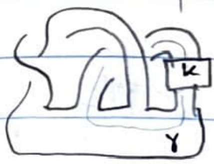
to explicitly see:



Cap off w/ two parallel copies of slice disks for K

★ these copies of K link 0 times

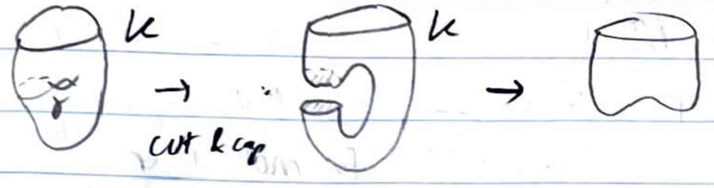
- another way:



$lk(\gamma, \gamma^+) = 0$

genus 1

cut along γ, cap off resulting boundary w/ two parallel copies of slice disk for K.



★ make sure multiple band moves don't create genus

★ ensure band moves are oriented (for an oriented surface)

- remark: {K, mK, K^r, mK^r} could be 1, 2, 4 knots up to isotopy or 1, 2, 4 concordance classes

↳ ★ If a genus one Seifert surface for K contains an embedded slice knot (γ in the picture) w/ surface framing zero, homologically essential, then K is slice

- def: recall algebraically slice knot K has metabolic Seifert form.

A geometric realization of a basis for the metabolizer is a derivative of K
↳ part that vanishes

e.g.

$$V = \begin{array}{c|cc} & x^+ & y^+ \\ \hline x & -1 & 1 \\ y & 0 & 0 \end{array}$$

y is an example of a derivative
x+y also is a derivative

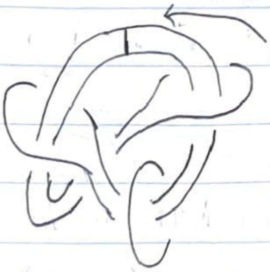
★ finding surface framing zero curves

- e.g.:

or

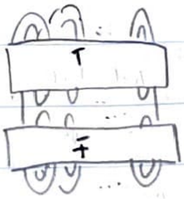
* surface framing is zero

- def: an n-component link L is strongly slice if L bounds n disjoint disks in B^4.
- e.g.: O O O
- e.g.:



do a band move here

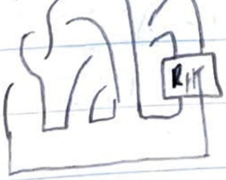
- recall: K slice ⇒ K algebraically slice (converse not true)
- K alg. slice w/ a Seifert surface of genus g w/ g-component slice link representing basis for metabolizer ⇒ K slice
- e.g.: K#-K slice



e.g.:

genus 2 surface w/ 2 component slice link representing basis for metabolizer

- e.g.: algebraically (and in fact top. slice) slice knot but not sm. slice Wh(RHT)



(continue')

→ - Kauffman's conjecture: If K is a slice knot w/ a genus 1 Seifert surface F, then $\exists$ an essential simple closed curve d on F s.t. $lk(d, d^+) = 0$, d slice knot
  ↳ False, disproved Cochran-Davis (2013)
- e.g. "infection along a curve"

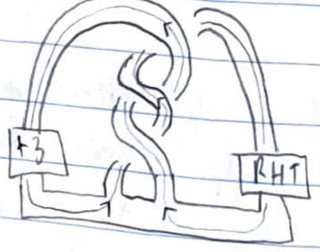
$P_\eta(K) =$

equivalently, union of complement of η with a nbhd of K,
meridian of η ↔ 0-framed longitude of K
0-framed long. of η ↔ meridian of K

- using curves on Seifert surfaces to build slice disks

e.g.:

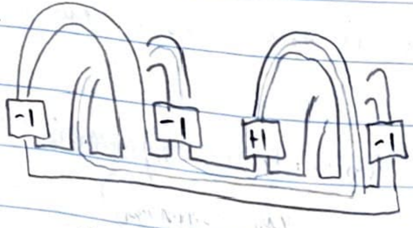
$K =$



surface framing zero ✓
the curve is LHT # RHT, is slice
so can cut along curve and glue 2 parallel copies of slice disk for LHT#RHT to cap off.

$\Rightarrow K$ slice

e.g.



exercise: $K = RHT \# 4_1$
$\sigma(K) = -2 \Rightarrow K$ not top slice $\Rightarrow 1 \leq g_4^{top}(K)$

cut along this curve and glue two slice disks to reduce genus by 1
$\Rightarrow g_4^{top}(K) = g_4^{sm}(K) = 1$.
(construct genus 1 surface)

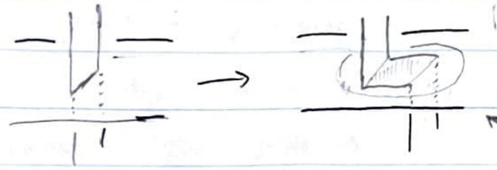
- exercise: $g_4^{sm}((2,1)\text{-cable of } 4_1) \leq 1$

known to be not ribbon
only recently proved to not be slice

- def: a Seifert surface is excellent if it admits a derivative that is a trivial link
- exercise: $J \# -J$ admits an excellent Seifert surface
- note: knots which admit excellent Seifert surfaces are sm. slice.

- **Prop:** (Cochran-Davis) (slice ⇒ ribbon) ⟺ (every slice knot has an excellent Seifert surface)

Pf: (⇒): [WTS ribbon knots have excellent Seifert surfaces]



immersed ribbon disk ⇝ Seifert surface

Claim: this Seifert surface is perfect

Pf: go around the hole □

(⇐): [WTS: K w/ excellent Seifert surface is ribbon]

recall: ribbon disks can be described as a n-comp unlink and $n-1$ bands.

Suppose K has excellent genus g Seifert surface F and g component unlink L. Take 2 antiparallel copies of L and attach $2g-1$ bands to obtain immersed ribbon disk for K

antiparallel: O O O … ⇝ ◎ ◎ … [1/29 pic] □

- exercise: in general these constructions are not inverse to each other
- concordance invariants
  - Arf
  - Fox-Milnor
  - signature
  - Levine-Tristram signature
  - Rasmussen s-invariant (via perturbation of Khovanov homology)
  - Ozsváth-Szabó τ-invariant (knot Floer homology)
- note: concordance invariants from Khovanov & Floer homology are invariants of smooth knot concordance, as opposed to invariants from Seifert form which are invariants of topological knot concordance.
  - ↳ $\tau, s: \mathcal{C}_{sm} \to \mathbb{Z}$ don't factor through $\mathcal{C}$ top.

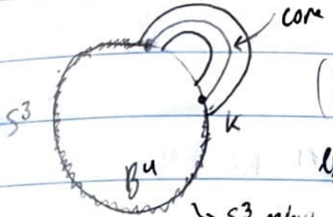
Knots $\rightsquigarrow$ 4-manifolds

- def: the n-trace of a knot K in S^3 is $X_n(K) := B^4 \cup$ n-framed 2-handle attached along $K \subset \partial B^4$ $= B^4 \cup_\varphi (D^2 \times D^2)$ $\quad \partial(D^2 \times D^2) = \partial D^2 \times D^2 \sqcup D^2 \times \partial D^2$

  $\varphi : \partial D^2 \times D^2 \to \nu(K)$ ↳ glued along first factor

  $x \in \partial D^2 \qquad S^1 \times \{x\} \mapsto$ n-framed longitude of K

- note: $X_n(K)$ has boundary



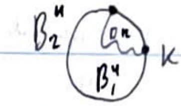
$(S^3 - \nu(K)) \cup (D^2 \times S^1)$

exercise: use φ to conclude that $\partial(X_n(K)) = S^3_n(K)$

↳ S^3 minus nbhd of knot $\qquad$ n-surgery of K

- ex: $X_0(U) = S^2 \times D^2$
- Trace Embedding Lemma: The 0-trace $X_0(K)$ admits a smooth (locally collared) embedding into S^4 iff K is smooth (topologically) slice.

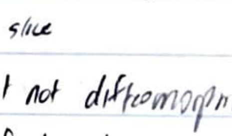
  pf: ($\Leftarrow$): Sps K slice, consider $S^4 = B^4_1 \cup_{S^3} B^4_2$.



  Then $X_0(K) \cong B^4_2 \cup \nu(D_K)$ is the desired embedding

  exercise: this is a 0-framed 2-handle

  ($\Rightarrow$): $\varphi : S^2 \to X_0(K)$, $\varphi(S^2) =$ core of 2-handle $\cup$ cone on K



  φ sm away from cone pt. p. Let $i : X_0(K) \to S^4$ be the assumed embedding, $i \circ \varphi : S^2 \to S^4$ is sm (locally flat) away from $i(p)$. So $i \circ \varphi|_{S^2 \setminus \nu(\varphi^{-1}(p))} \cong D^2$ is sm (locally flat) embedding of D^2 into $S^4 - \nu(i(p)) \cong B^4$ $\square$

- Cor: $X_0(K)$ sm (locally collared) embeds into $\mathbb{R}^4$ iff K is sm (top) slice
- def: M_1, M_2 are an exotic pair if they are homeomorphic but not diffeomorphic
- thm (Freedman-Quinn): Let M connected noncompact 4-mfd. If desired, fix a sm str on any collection of connected comps of ∂M. Then $\exists$ sm str on M extending the given sm str on (a subset of) ∂M.
- Let K be a top. slice knot that is not sm. slice (e.g. Wh(RHT)). By Trace Embedding Lemma, $\exists$ locally collared embedding $i : X_0(K) \to \mathbb{R}^4$. Then $i(X_0(K))$ inherits a sm. str from $X_0(K)$. Since i is locally collared, $\mathbb{R}^4 - \text{int}(i(X_0(K)))$ is a mfd. Also it is connected & noncompact. Freedman-Quinn $\Rightarrow$ extend sm str. on $i(\partial X_0(K))$ to the rest of

$\mathbb{R}^4 - \text{int}(i(X_0(K)))$ giving a sm. mfd $\mathcal{R}$ homeomorphic to $\mathbb{R}^4$.
We claim $\mathcal{R}$ is not diffeo to $\mathbb{R}^4$. Sps it was, then
we have a sm embedding $X_0 \to \mathcal{R} \overset{\text{diff}}{\cong} \mathbb{R}^4$. By Cor to
Trace Embedding Lemma, K is sm slice ↯ □

- C = Conway knot, KT = Kinoshita - Terasaka knot (differs by mutation)
- exercise: 1) $\Delta_C(t) = \Delta_{KT}(t) = 1$ (so both top. slice)

  2) KT is ribbon (so is sm. slice)
- Q: is C sm. slice? (all known sm. conc. invariants vanish on C)

  A: No (Lisa Piccirillo)

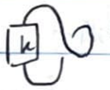
  Pf Sketch: 1) Build a knot J w/ $X_0(J) \cong X_0(C)$

  2) Show J is not sm slice (using Rasmussen s-invariant)

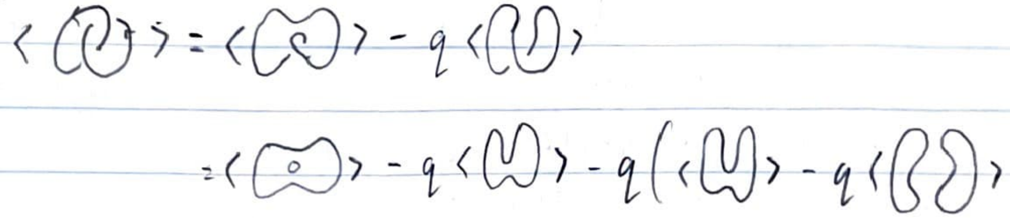
  3) T.E.L $\Rightarrow X_0(J)$ doesn't sm. embed into S^4 so $X_0(C)$ doesn't sm. embed, so C not slice □

Khovanov homology & s-invariant

[Melissa Zhang notes on Khovanov homology]

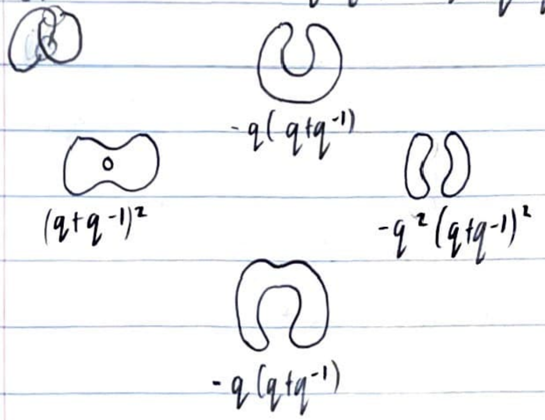
- def the <u>Kauffman bracket</u> $\langle D \rangle$ of a link diagram D is the Laurent poly in q defined recursively by the rules
  1) $\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$
  2) $\langle L \sqcup O \rangle = (q+q^{-1}) \langle L \rangle$
  3) $\langle \emptyset \rangle = 1$
- e.g.: $\langle O \rangle = q+q^{-1}$
  $\langle O \; O \rangle = (q+q^{-1})^2$, $\langle O \ldots O \rangle = (q+q^{-1})^n$
  ⟨⟩ = ⟨⟩ $- q \langle O O \rangle = q+q^{-1} - q(q+q^{-1})^2$
- Note: Kauffman bracket is <u>not</u> a link invariant
- exercise: $D =$  $D' =$  $\langle D \rangle = -q^2 \langle D' \rangle$

- def: let D be a diagram w/ n_- negative crossings and n_+ positive crossings. The <u>(unnormalized) Jones poly</u> of D is $\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$
- exercise: $\hat{J}$ is a link invariant
  (Check how Reidemeister moves change)
- def: The <u>Jones poly</u> is $J(D) = \hat{J}(D)/_{q+q^{-1}}$ $(J(O) = 1)$
- e.g. $D =$ 



$= (q+q^{-1})^2 - q(q+q^{-1}) - q((q+q^{-1}) - q(q+q^{-1})^2)$

OR



$n_- = 0, n_+ = 2$

$\hat{J}(D) = (-1)^0 q^2 (\langle D \rangle)$

$= q^6 + q^4 + q^2 + 1$

- graded Euler characteristic: $\chi_q(C) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \operatorname{rank}(H^{i,j}(C))$
- $V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ bigraded $\mathbb{Z}$-module $v_\pm$ bigrading $(0, \pm 1)$
- note: $\chi_q(V) = q + q^{-1}$
- notation: $C = \bigoplus_{i,j} C_{i,j}$ $C[n]\{m\}_{i,j} = C_{i-n,\,j-m}$
- e.g.: $V = \begin{array}{c} \mathbb{Z}v_+ \\ \hline \mathbb{Z}v_- \end{array}$ $V[2]\{3\} =$

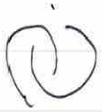
- e.g.: $V \otimes V =$ $v_+ \otimes v_+$

$v_- \otimes v_+ \quad v_+ \otimes v_-$

$v_- \otimes v_-$

$V \otimes V\{2\}$

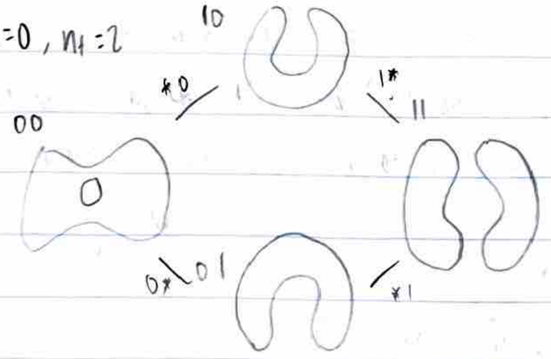
$\alpha \in [0,1]^n$



$k_\alpha = \#$ of circles $\cdot$ $V_\alpha = V^{\otimes k_\alpha}[n_-]\{r_\alpha + n_+ - 2n_-\}$

$\uparrow$ # 1's in α

$n_- = 0,\ n_+ = 2$



$V[0]\{3\}$

$V \otimes V[0]\{2\}$ $V \otimes V[0]\{4\}$

$V[0]\{3\}$

h= 0 1 2

bigrading (h,q)

q / h	0	1	2	χ
6			•	$0-0+1=1$
4	•	• •	• •	$1-2+2=1$
2	• •	• •	•	$2-2+1=1$
0	•			$1-0+0=1$

$\Big\}\ q^6 + q^4 + q^2 + 1$

$\hookrightarrow$ exercise: graded Euler char gives $\hat{J}(D)$

differential on $CKh(D)$

- along edges of cube, circles either merge or split

merge: $m: V \otimes V \to V$

$v_+ \otimes v_+ \mapsto v_+$

$v_+ \otimes v_-,\ v_- \otimes v_+ \mapsto v_-$

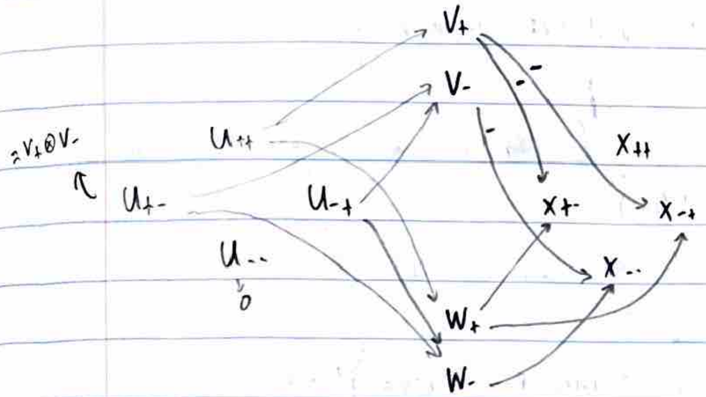
$v_- \otimes v_- \mapsto 0$

split: $\Delta: V \to V \otimes V$

$v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+$

$v_- \mapsto v_- \otimes v_-$

- differential is id on passive circles along an edge
- ± signs needed along edges according to # of 1's before * in edge label (for $d^2=0$)



What survives in homology:

q/h	0	1	2	χ
6			$[x_{++}]$	1
4			$[x_{+-}]=[-x_{-+}]$	1
2	$[u_{+-}-u_{-+}]$			1
0	$[u_{--}]$			1

- exercise: Compute Khovanov homology of trefoil (8 vertices on cube)
- thm: Khovanov homology is a link invariant (Pf Reidemeister moves)

Idea: C be a chain complex, $A \subseteq C$ subcomplex (A is a submodule, $dA \subseteq A$). then we have a short exact sequence $0 \to A \to C \to C/A \to 0$

- Lemma: Let $0 \to A \to C \to C/A \to 0$ be a s.e.s. of chain complexes
  1) If $A \simeq 0$ (chain homotopy equivalence) then $C \simeq C/A$
  2) If $C/A \simeq 0$, then $A \simeq C$ [A acyclic means $A \simeq 0$ ("no cycles")]

find sub or quotient complexes that are $\simeq 0$.

- Lee perturbation: changing differential map $d_{Lee} = d_{Kh} + \Phi$ $C_{Lee}(L)$

Merge: $v_- \otimes v_- \mapsto 0 \; (+ v_+)$ split: $v_- \mapsto v_- \otimes v_- + v_+ \otimes v_+$

↳ does not preserve quantum grading exercise: $d_{Lee}^2 = 0$

- def: let (C,d) be a chain complex. A filtration on (C,d) is a sequence of subcomplexes $\ldots \supseteq F_i \supseteq F_{i+1} \supseteq \ldots$ s.t. $\bigcap_i F_i = \emptyset$, $\bigcup_i F_i = C$

↳ generally interested in finite length filtrations where only finitely many F_i are not $\emptyset$ or C: $C = F_n \supseteq F_{n+1} \supseteq \ldots \supseteq F_{n+k} = \emptyset$

↳ differential needs to land inside ($dF_j \subseteq F_j$)

- note: (CKh, d_{Lee}) is a filtered chain complex w/ $F_i = \bigoplus_{q \geq i} CKh_q$
- note: Lee complex has a well-def $\mathbb{Z}_4$ quantum grading

- For working w/ S we will work w/ $\mathbb{Q}$ coefficients (works w/ field of any characteristic) : $V = \mathbb{Q}_{v_+} \oplus \mathbb{Q}v_-$
- Consider change of basis:

$a = v_- + v_+$ $\qquad$ $b = v_- - v_+$

- note: 1) these elts don't have a well-def quantum grading as they are not homogeneously graded

2) We can still consider the quantum filtration of a nonhomogeneous graded elt x

$gr_q(x) = \max\{i \mid x \in F^i(C_{Lee})\}$ $\qquad$ $(\supseteq_{\dots} F_i \supseteq F_{i+1} \supseteq \dots \supseteq \emptyset$

and the quantum filtration of a homology class [X]:

$gr_q([X]) = \max\{gr_q(y) \mid [y] = [X]\}$

- exercise: $m' : a \otimes a \mapsto 2a,\ a \otimes b, b \otimes a \mapsto 0,\ b \otimes b \mapsto -2b$

$\Delta' : a \mapsto a \otimes a,\ b \mapsto b \otimes b$

- thm: the Lee homology of an n-comp link L is $Lee(L) \cong (\mathbb{Q} \oplus \mathbb{Q})^n$

↳ corresponds to 2^n orientations of the link

- Lee's canonical generators:

D :  checkerboard coloring

leave infinite region unshaded

D_0 :  oriented resolution

draw a dot to left of any pt on each circle

$\boxed{s_0 = a \otimes b}$ if a dot is in a sh (resp. unsh) region, label it a (resp. b

- exercise: $s_0 \in \ker d_{Lee}$

Let K be a knot. ↗ q-grading is smallest q in homogeneous basis

$s_{min}(K) = \min\{gr_q([X]) : [X] \in Lee(K), [X] \neq 0\}$

$s_{max}(K) = \max\{ \quad \dots \quad \}$

- def (Rasmussen): $s(K) = (s_{min}(K) + s_{max}(K))/2$ $\quad$ * always even b/c $s_{min/max}$ odd
- Prop: $s_{max} = s_{min} + 2$ (so $s(K) = s_{min}(K) + 1 = s_{max} - 1$)
- exercise: for a knot K, quantum gradings are odd
- $C_{Lee}(K)$ has $\mathbb{Z}_4$ quantum grading $\Rightarrow C_{Lee}(K) \cong C_{Lee,1}(K) \oplus C_{Lee,-1}(K)$

where $C_{Lee,\pm1}(K)$ is the summand in $\mathbb{Z}_4$-quantum-grading ±1

(goal: prove prop)

- <u>Lemma</u>: let D be a diagram for a knot K, O an orientation. Then s_o + $s_{\bar{o}}$ are in two different $\mathbb{Z}_4$-quantum gradings of $C_{Lee}(D)$. (Pf: exercise)

So $s_o \pm s_{\bar{o}}$ generate the summands in $Lee(K) \cong Lee_1(K) \oplus Lee_{-1}(K)$

$\Rightarrow$ $s_{max} \neq s_{min}$ b/c Lee homology is supported in at least 2 gradings

$\Rightarrow$ $s_{min}(K) = gr_q([s_o]) = gr_q([s_{\bar{o}}])$ since both $[s_o], [s_{\bar{o}}]$ have components in $\mathbb{Z}_4$-quantum grading $s_{min}(K)$

<u>Algebraic detour: mapping cone</u>

Let $f: A \to B$ be a chain map between $A = (\bigoplus_i A^i, d_A)$, $B = (\bigoplus_i B^i, d_B)$

- the <u>mapping cone of f</u> is $C(f) := \left(\bigoplus_i (A^{i+1} \oplus B^i), \bigoplus_i \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}\right)$

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r, "-d_A"] \arrow[d, "f"] & A^0 \arrow[r, "-d_A"] \arrow[d, "f"] & A^1 \arrow[r, "-d_A"] \arrow[d, "f"] & A^2 \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r, "d_B"'] & B^0 \arrow[r, "d_B"'] & B^1 \arrow[r, "d_B"'] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

note/exercise: $d^2_{C(f)} = 0$

in practice, write $C(f) = (A \xrightarrow{f} B)$

- <u>note</u>: $\exists$ s.e.s $0 \to B \to C(f) \to A[-1] \to 0$

- <u>ex</u>: ∞, $m: OO \to \infty$, $V \otimes V \to V$ $\quad CKh(\infty) = C(m: OO \to \infty \overset{\text{to fix quantum grading}}{\{1\}})$

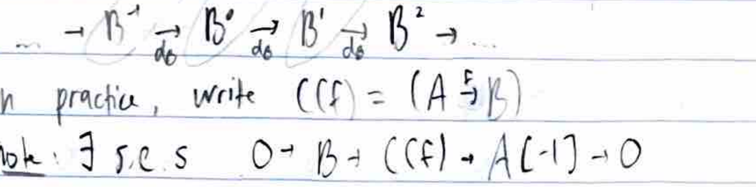
- <u>def</u>: A chain map $f: C \to C'$ between filtered chain complexes is <u>filtered of deg. k</u> if $f(F_i) \subseteq F_{i+k}$

- <u>Lemma</u> Let K_1, K_2 be knots. Then $\exists$ s.e.s.

$$0 \to Lee(K_1 \# K_2^r) \xrightarrow{p_*} Lee(K_1) \otimes Lee(K_2) \xrightarrow{m_*} Lee(K_1 \# K_2) \to 0$$

where m_*, p_* have filtered degree -1

Pf: Let D_1, D_2 be diagrams for K_1, K_2



$K_1 \# K_2^r$ $\qquad$ $K_1 \sqcup K_2$ $\qquad$ $K_1 \# K_2$

s.e.s: $0 \to C_{Lee}(K_1 \# K_2)\{1\} \xrightarrow{i} C_{Lee}(K_1 \# K_2^r) \xrightarrow{p} C_{Lee}(K_1 \sqcup K_2) \to 0$

inclusion $\qquad$ projection

induces l.e.s on homology

$$\begin{tikzcd}
\overset{\dim 2,\ \mathbb{Q}\oplus\mathbb{Q}}{Lee(K_1 \# K_2)} \arrow[rr, "i_*"] & & \overset{\dim 2,\ \mathbb{Q}\oplus\mathbb{Q}}{Lee(K_1 \# K_2')} \arrow[dl, "p_*"] \\
& \underset{\dim 4,\ (\mathbb{Q}\oplus\mathbb{Q})^2 = \mathbb{Q}^4}{Lee(K_1 \sqcup K_2)} \arrow[ul] &
\end{tikzcd}$$

exercise: $i_* = 0$ (b/c $2+2=4$)

so l.e.s splits:

$0 \to Lee(K_1 \# K_2') \xrightarrow{p_*} Lee(K_1 \sqcup K_2) \xrightarrow{m_*} Lee(K_1 \# K_2) \to 0$

exercise: p_*, m_* filtered of deg -1. $\square$

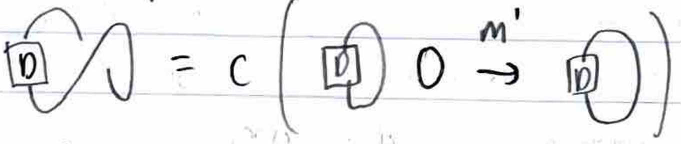
- Pf of Prop: Above lemma w/ $K_1 = K$, $K_2 = U$

$0 \to Lee(K) \xrightarrow{p_*} Lee(K) \otimes Lee(U) \xrightarrow{m_*} Lee(K) \to 0$

$Lee(U) = \mathbb{Q}a \oplus \mathbb{Q}b$ $\quad$ p_*, m_* filtered of deg -1

$\uparrow$ generated by a

One of $\{s_o, s_{\bar{o}}\}$ has a label 'a' on component where connected sum occurs, call this generator s_a and the other s_b



$gr_q([s_a + \varepsilon s_b]) = s_{max}$ for $\varepsilon = 1$ or -1

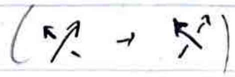
$m^*([s_a + \varepsilon s_b] \otimes a) = [s_a]$ by def of m

since m^* filtered of deg -1, $gr_q([s_a]) \geq gr_q([s_a + \varepsilon s_b] \otimes a) - 1$

$\Rightarrow gr_q([s_a + \varepsilon s_b] \otimes a) \leq gr_q([s_a]) + 1$ $\quad \swarrow \otimes a$ gives -1 shift

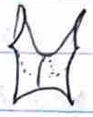
since $s_{min}(K) = gr_q([s_o]) = gr_q([s_{\bar{o}}])$ $\quad s_{max}(K) - 1 \leq s_{min}(K) + 1$

but $s_{max} \neq s_{min}$ so $s_{max}(K) = s_{min}(K) + 2$ $\square$

- exercise: Show $s(RHT) = 2$
- Properties of s:
  - exercise: Sps K_+, K_- differ by a single crossing  then $s(K_-) \leq s(K_+) \leq s(K_-) + 2$
  - $s(\text{unknot}) = 0$
  - exercise: $s(-K) = -s(K)$
  - exercise: $s(K_1 \# K_2) = s(K_1) + s(K_2)$
  - thm (Rasmussen): $|s(K)/2| \leq g_4^{sm}(K)$

Pf idea: view minimal genus surface for K as a genus $g_4^{sm}(K)$ cobordism b/w U and K. decompose cobordism into elt. cobordisms

minima maxima saddles

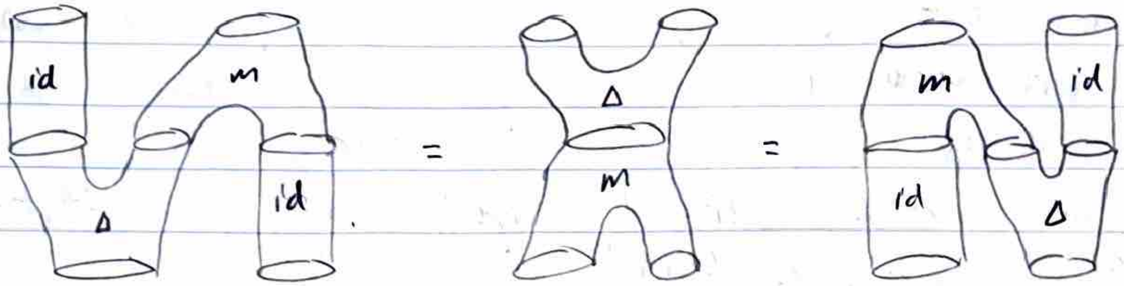


cup cap merge/split

Khovanov homology as (1+1)-TQFT

looking at S^1 (knots +1 dim surfaces

- def: A Frobenius system is the data $(R, A, \iota, m, \varepsilon, \Delta)$
  1) commutative ground ring (e.g. $\mathbb{Z}$, $\mathbb{Q}$)
  2) R-algebra A, in particular
     a) inclusion map $\iota: R \to A$, $1 \mapsto 1$
     b) multiplication map $m: A \otimes A \to A$ "(merge map)"
  3) comultiplication map $\Delta: A \to A \otimes A$ that is "(split map)"
     coassociative and cocommutative i.e
$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, phantom, "O"] & A \otimes A \arrow[d, "\Delta \otimes id"] \\ A \otimes A \arrow[r, "id \otimes \Delta"'] & A \otimes A \otimes A \end{tikzcd}$$
  4) R-module co-unit $\varepsilon: A \to R$
     s.t. $(\varepsilon \otimes id) \circ \Delta = id$
- def: A is a Frobenius algebra i.e it is both an algebra and a coalg and the following holds: $(id_A \otimes m) \circ (\Delta \otimes id_A) = \Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta)$



cobordisms go up

- ex: $(\mathbb{Z}, V, \iota, m, \varepsilon, \Delta)$ (Khovanov homology) is a Frobenius system
  $\iota: \mathbb{Z} \to V$ $1 \mapsto V_+$ $m: V_+ \otimes V_+ \mapsto V_+$ ← V_+ "looks like" id
  $\varepsilon: V \to \mathbb{Z}$ $V_+ \mapsto 0$, $V_- \mapsto 1$ $V_+ \otimes V_-, V_- \otimes V_+ \mapsto V_-$ ↙
  $V_- \otimes V_- \mapsto 0$
- also Lee perturbation gives Frob. system

key idea: a cobordism $F: K_0 \to K_1$ induces chain map

$CKh(F): CKh(K_0) \to CKh(K_1)$

$CLee(F): CLee(K_0) \to CLee(K_1)$

$CLee(F)$ is a filtered map of deg $\chi(F)$ and this map is nonzero $\square$

- Corollary: $\frac{s}{2}: \mathcal{C} \to \mathbb{Z}$ is a surjective homomorphism
- thm: If D is positive diagram for a positive knot K (all crossings are positive) then $s(K) = gr_q(s_o)+1$
- exercise: Show $s(T_{p,q}) = (p-1)(q-1)$ ($T_{p,q}$ is a pos. knot)

* at some point in your mathematical career you stop caring abt normalization (+/., const fac.)

and conclude $g_4^{sm}(T_{p,q}) = (p-1)(q-1)(\frac{1}{2}) = U(T_{p,q})$

- exercise: $g_4^{sm}(K) \le U(K)$ (unknotting #)

↳ crossing change gives a genus 1 cobordism

- exercise (hard maybe?): If K is alternating then $s(K) = -\sigma(K)$

↳ maybe use checkerboard coloring def of σ

- def: a slice-torus invariant is a homomorphism $\varphi: \mathcal{C} \to \mathbb{R}$ s.t.
  1) (slice) $\forall K$, $\varphi(K) \le g_4(K)$
  2) (torus) $\forall p,q > 0$ coprime, $\varphi(T_{p,q}) = g_4(T_{p,q})$

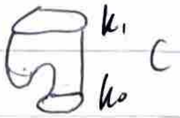
e.g.: $s/2$ is a slice-torus invariant

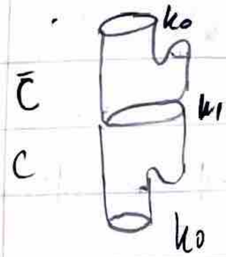
τ (coming from Heegaard-Floer homology) is also a slice-torus inv.

s and τ have a lot in common but aren't the same

($\frac{s}{2} \oplus \tau : \mathcal{C} \to \mathbb{Z} \oplus \mathbb{Z}$ is surjective)

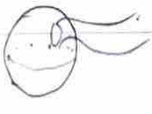
- thm: (Levine-Zemke): If $C: K_0 \to K_1$ is a ribbon concordance (no maxima, direction matters, concordance going up) then $Kh(C): Kh(K_0) \to Kh(K_1)$ is injective, with left inverse $Kh(\bar{C})$.





($\bar{C}$ is C upside down, opposite orientation)

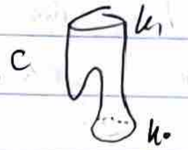
↑ the entire is id on Kh



- <u>key lemma</u> (Zemke): Let $C: k_0 \to k_1$ be a ribbon concordance with n births, n saddles, $\bar{C}: k_1 \to k_0$ upsidedown. Then $\bar{C} \circ C: k_0 \to k_0$ is isotopic to $k_0 \times I$ with n unknotted, unlinked 2-spheres tubed on.

* ribbon concordance induces nice maps on Khovanov homology: other invariants?

- <u>Prop</u> (Gordon): If $C: k_0 \to k_1$ is a ribbon concordance, $X = S^3 \times I \setminus \nu(C)$, $Y_i = S^3 \setminus \nu(k_i)$, then
  1) $\pi_1(Y_0) \hookrightarrow \pi_1(X)$ injective
  2) $\pi_1(Y_1) \twoheadrightarrow \pi_1(X)$ surjective



<u>Pf (2)</u>: k-handle added to C $\rightsquigarrow$ $(k+1)$-handle added to X

$X = (Y_0 \times I) \cup$ 1-handles $\cup$ 2-handles or dually,

$X = (Y_1 \times I) \cup$ 2-handles $\cup$ 3-handles

$\Rightarrow \pi_1(Y_1) \to \pi_1(X)$ surjective (2-handles add relations, 3-handles do nothing)

(1): <u>exercise</u>: $i_*: H_*(Y_0) \to H_*(Y_1)$ isomorphism.

: In $X = (Y_0) \times I \cup$ 1-handle $\cup$ 2-handle, # 1-handles = # 2-handles $(=n)$ and 2-handles must cancel 1-handles homologically

Van Kampen $\Rightarrow$ $\pi_1(X) = \langle \pi_1(Y_0) * F \rangle / \langle r_1, \ldots, r_n \rangle$, $F = \langle x_1, \ldots, x_n \rangle$ free gp on n

$\varepsilon_i(r_j)$ = exponent of x_i in r_j, $(\varepsilon_i(r_j))$ $n \times n$ matrix, has det ± 1 (from 2-handles cancelling 1-handles)

- <u>def</u>: A group G is <u>residually finite</u> if $\forall g \in G$, $g \neq 1$, $\exists$ homomorphism $h: G \to$ finite gp s.t. $h(g) \neq 1$
- <u>Prop</u> (Thurston): $\pi_1(Y_0)$ is residually finite.

Let $z \in \ker(\pi_1(Y_0) \to \pi_1(X))$. If $z \neq 1$ then by residual finiteness, $\exists p: \pi_1(Y_0) \to G$, $|G| < \infty$, s.t. $p(z) \neq 1$. Then:

$\pi_1(X) \to \underbrace{(G * F) / \langle p'(r_1), \ldots, p'(r_n) \rangle}_{H}$ where $p': \pi_1(Y_0) * F \to G * F$ induced by p.

$z \longmapsto 1$

group theory result shows $G \to H$ injective.

$$\begin{tikzcd}
z \arrow[d, "p"] & \pi_1(Y_0) \arrow[r] \arrow[d] \arrow[rd, phantom, "\circ"] & \pi_1(X) \arrow[d] \\
\neq 1 & G \arrow[r, hook] & H
\end{tikzcd}$$

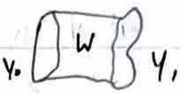
contradiction, so ker is trivial $\square$

- <u>def</u>: A <u>homotopy ribbon concordance</u> $C: K_0 \to K_1$ is a locally flat concordance st. 1) $\pi_1(Y_0) \hookrightarrow \pi_1(X)$ 2) $\pi_1(Y_1) \twoheadrightarrow \pi_1(X)$
  - ↳ we write $K_0 \leq_{top} K_1$ if $\exists$ homotopy ribbon concordance $K_0 \to K_1$
  - ↳ $K_0 \leq K_1$ if $\exists$ ribbon concordance $K_0 \to K_1$
- $K_0 \leq K_1 \Rightarrow K_0 \leq_{top} K_1$
- <u>thm</u> (Agol 2022): Ribbon concordance is a partial order.
  - ↳ hard part is antisymmetry: $K_0 \leq K_1$ and $K_1 \leq K_0 \Rightarrow K_0 = K_1$

Homology Cobordism

3-mfds are closed (compact w/o boundary), connected, oriented

- def: two 3-mfds Y_0, Y_1 are cobordant if $\exists$ sm compact 4-mfd W st. $\partial W = -Y_0 \sqcup Y_1$



  ↳ is an equivalence relation

- Prop: Every 3-mfd bounds a sm compact 4-mfd

  Pf: By Lickorish-Wallace, every 3-mfd Y is integral surgery on a link L in S^3. Let X be the 4-mfd obtained by attaching framed 2-handles along $L \subset \partial B^4$, then $\partial X = Y$. □

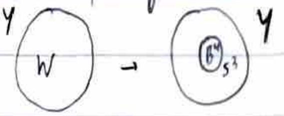
- Cor: Any two 3-mfds are cobordant (remove some ball so all cobordant to S^3)
- def: two 3-mfds Y_0, Y_1 are $\mathbb{Z}$-homology cobordant if $\exists$ sm compact 4-mfd W s.t. 1) $\partial W = -Y_0 \sqcup Y_1$ 2) $i_*: H_*(Y_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ isomorphism

  "W looks like a product in homology"

  ↳ $\mathbb{Z}$ can be replaced w/ any ring (e.g. $\mathbb{Q}$, $\mathbb{Z}_p$)

  ↳ is an equivalence relation

- e.g: Y any 3-mfd, then $Y \times I$ is an R-homology cobordism for any ring R
- e.g: Y bounds an $\mathbb{Z}H_*B^4$ W (integer homology ball) $\Leftrightarrow Y \underset{\text{cob}}{\overset{\sim}{\mathbb{Z}H_*}} S^3$ (works for any ring) [remove a ball]



- exercise: $K_0 \underset{sm}{\sim} K_1$ then $S^3_{p/q}(K_0) \underset{\text{cob}}{\overset{\sim}{\mathbb{Z}H_*}} S^3_{p/q}(K_1)$

  ↳ $S^3_{p/q}(K) = S^3 - \nu(K) \cup_\varphi (S^1 \times D^2)$ so $p\mu + q\lambda$ bounds a disk in $S^1 \times D^2$ meridean longitude

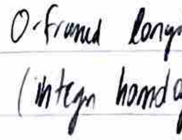
- note: 1) $H_1(S^3_{p/q}(K);\mathbb{Z}) = \mathbb{Z}_p$

  in $\nu(K)$, meridean generates H_1, can think of gluing in torus in 2 steps, first $p\mu + q\lambda$ then a ball (which doesn't affect H_1). 0-framed longitude is nullhomologous

  2) If $p = \pm 1$ then $S^3_{p/q}(K)$ is a $\mathbb{Z}HS^3$ (integer homology sphere)

  3) If $q \neq 0$ then $S^3_{p/q}(K)$ is a $\mathbb{Q}HS^3$

  Idea: surger the concordance to build W



- exercise: If K sm. slice then $\Sigma_q(K)$, $q = p^n$ for some prime p, bounds a $\mathbb{Q}HB^4$

  ↳ $\Sigma_q(K)$ q-fold cyclic branched cover of K

  $\Sigma_q(K) = (q\text{-fold cyclic cover of } S^3 \setminus \nu(K)) \cup (S^1 \times D^2)$

  $\pi_1(S^3 \setminus \nu(K)) = \mathbb{Z}$, use homo $\mathbb{Z} \to \mathbb{Z}_q$

$\Sigma_q(K)$ preimage of K is $S^1 \times \{0\}$

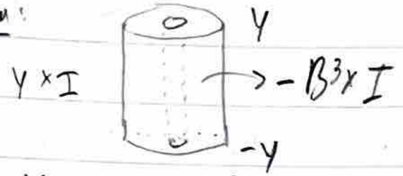
$\downarrow \pi$ π is modeled on $z \mapsto z^q$

S^3 along pt × D^2 (in $\mathbb{C}$)

<u>Idea</u>: take q-fold cyclic branched cover of B^4 branched over slice disk

- <u>exercise</u>: let Y be RHS^3. then $Y \# -Y$ bounds an RHB^4

  <u>Idea</u>:



  $W = (Y - B^3) \times I$

  $\partial W = Y \# -Y$

  check how domain homology

- <u>note</u>: Y_1, Y_2 $RHS^3 \Rightarrow Y_1 \# Y_2$ RHS^3

  $R = \mathbb{Z}$

- Consider $(\{\mathbb{Z}HS^3\}, \#)$

  Q: Y $\mathbb{Z}HS^3$, $Y \neq S^3$, $\exists Y'$ st $Y \# Y' = S^3$? A: No

- <u>def</u>: a <u>Heegaard splitting</u> of a 3-mfd Y is a decomposition $Y = H_1 \cup H_2$ where H_1, H_2 are handlebodies of genus g and $\varphi: \partial H_1 \to \partial H_2$ is an orientation reversing homeomorphism. g is genus of Heegaard splitting, $\partial H_1 \cong -\partial H_2$ is <u>Heegaard surface</u>

- <u>e.g.</u> $S^3 = B^3 \cup B^3$

  <u>e.g.</u> $S^1 \times S^2 = (S^1 \times D^2) \cup_\varphi (S^1 \times D^2)$ $\varphi = -\mathrm{id}_{S^1 \times D^2}$

  for each $x \in S^1$, the D^2's are glued to give a S^2

  <u>e.g.</u>: $L(p,q) = (S^1 \times D^2) \cup_\varphi (S^1 \times D^2)$ for some φ. <u>exercise</u>: find φ

  <u>e.g.</u>: $S^3 = (S^1 \times D^2) \cup_\varphi (S^1 \times D^2)$ $\varphi: \lambda \mapsto \mu, \mu \mapsto \lambda$

- <u>def</u>: the <u>Heegaard genus</u> of a 3 mfd Y is the min genus over all Heegaard splittings

  ↳ S^3 is the only 3 mfd w/ Heegaard genus 0

  ↳ Heegaard genus of $S^1 \times S^2$, $L(p,q)$ is 1

- <u>thm</u> (Haken): Heegaard genus is additive under connect sum

  ↳ so if $Y \neq S^3$ (H. genus > 0), $\nexists Y'$ st. $Y \# Y' = S^3$

- <u>def</u>: the <u>$\mathbb{Z}$-homology cobordism gp</u> is $\Theta^3_{\mathbb{Z}} = (\{\mathbb{Z}HS^3\}/\sim_{\mathbb{Z}H_* \text{ cob}}, \#)$

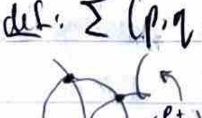
  ↳ id is $[S^3]$

  ↳ inverse of $[Y]$ is $[-Y]$

  ↳ this gp is nontrivial: see Rokhlin invariant μ

- <u>def</u>: $\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5_\varepsilon \subset \mathbb{C}^3$ (ε small).

  dim 4, codim 2   dim 5, codim 1



<u>Brieskorn homology sphere</u>

- Know: $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}^\infty$ surj homomorphism.

<u>Open</u>: $\exists$ nontrivial torsion in $\Theta^3_{\mathbb{Z}}$?

↳ 2-torsion: if $Y \cong -Y \Rightarrow [Y \# Y] = [S^3]$ in $\Theta^3_{\mathbb{Z}}$ but it is hard to show such a Y is nontrivial in $\Theta^3_{\mathbb{Z}}$, especially b/c:

<u>Thm</u>: If $\mu(Y) = 1$ then Y is not order 2 in $\Theta^3_{\mathbb{Z}}$ $([Y \# Y] \neq [S^3])$

<u>Idea</u>: homology cobordism invariant $\beta \in \mathbb{Z}$

1) $\beta(-Y) = -\beta(Y)$

2) $\beta(Y) \bmod 2 = \mu(Y)$

3) $Y_0 \underset{\text{cob}}{\sim} Y_1 \Rightarrow \beta(Y_0) = \beta(Y_1)$

$\mu(Y) = 1 \overset{(2)}{\Rightarrow} \beta(Y)$ odd

$\overset{(1)}{\Rightarrow} \beta(-Y) = -\beta(Y) \neq \beta(Y)$ ← $\beta(Y)$ nonzero

$\Rightarrow Y \underset{\mathbb{Z}HS\ \text{cob}}{\not\sim} -Y$

- <u>Thm</u>: $\exists$ nontriangulable n-dim top. mfds $\forall n \geq 5$

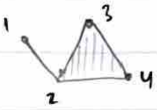
Triangulations

- <u>def</u>: A <u>simplicial complex</u> $K = (V, S)$ consists of V = finite collection of vertices S = finite collection of simplices (elts of $P(V)$) s.t. if $\sigma \in S$ and $\tau \subseteq \sigma$ then $\tau \in S$.

  this is an abstract simplicial complex. We can associate it:

- <u>def</u>: <u>geometric realization</u> of K: construct inductively for each $d \geq 0$, attach a d-simplex for each $\sigma \in S$ of $d = \text{card}(\sigma)$

  <u>e.g.</u> $V = \{1, 2, 3, 4\}$



$S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{2,4\}, \{2,3,4\}\}$

- <u>def</u>: the <u>closure</u> of a subset $S' \subseteq S$ is $Cl(S') = \{\tau \in S : \tau \subseteq \sigma \in S'\}$ (add in all the elts to make S' a simplicial complex)
- <u>def</u>: the <u>star</u> of a simplex $\tau \in S$ is $St(\tau) = \{\sigma \in S : \tau \subseteq \sigma\}$ (all the simplices containing τ)
- <u>def</u>: the <u>link</u> of a simplex $\tau \in S$ is $Lk(\tau) = \{\sigma \in Cl(St(\tau)) : \tau \cap \sigma = \emptyset\}$

  <u>e.g</u>: $St(\{3\}) = \{\{3\}, \{2,3\}, \{3,4\}, \{2,3,4\}\}$

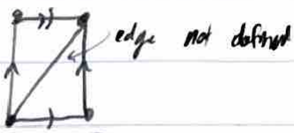
  $Cl(St(\{3\})) = \{\{3\}, \{2,3\}, \{3,4\}, \{2,3,4\}, \{2\}, \{4\}, \{2,4\}\}$

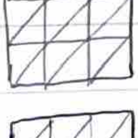
  $LK(\{3\}) = \{\{2\}, \{4\}, \{2,4\}\}$

- <u>def</u>: a triangulation of a topological space X is a homeomorphism from X to a simplicial complex

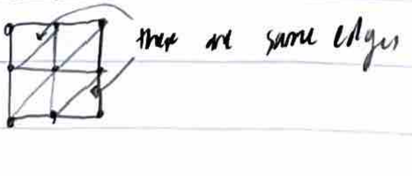
<u>e.g</u>:

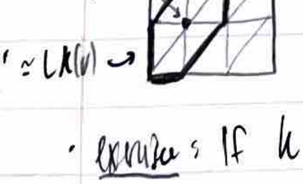
not triangulations:

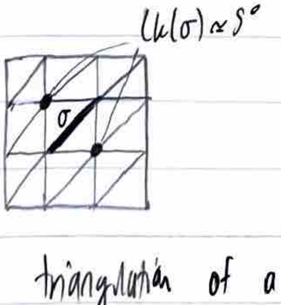




✓ is a triangulation

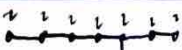






- <u>exercise</u>: If K is a triangulation of a top. mfd M^n, $\sigma \in K^{n-k}$, then $Lk(\sigma)$ is a $\mathbb{Z}H_*S^{k-1}$

- exercise: a triangulation on M induces a triangulation on its suspension ΣM, 

and lk of cone pt is M.
- Categories of mflds:
  - topological mflds: transition fns are continuous
  - PL mflds: transition fns are piecewise linear
  - smooth mflds: transition fns are C^∞
- def: A triangulation is combinatorial if the link of every simplex (equivalently, of every vertex) is PL-homeomorphic to a sphere
  ↳ if a space X admits a combinatorial triangulation, then X is a PL-mfd
  ↳ converse also true (so having combinatorial triang ⟺ PL-mfd)
- e.g.: Non PL triang of a top. mfd:
  Let P = homology sphere w/ nontrivial π_1 (e.g. Poincaré homology sphere).
  Fact 1: ΣP is not a mfd.
  Fact 2: (Double suspension theorem): $\Sigma(\Sigma P)$ is a top. mfd, homeomorphic to a sphere.
  So take triang on P, induces triangulation on $\Sigma^2 P$, but this triang is not combinatorial: Lk(cone pt) is ΣP which is not a mfd so not PL-homeo to a sphere.
  [$\Sigma^2 P$ does admit a PL-structure, just the one we constructed, not PL]
- Q (Poincaré 1899): does every sm mfd admit a triangulation?
  A: Yes, every sm mfd has a PL-structure, so has a combinatorial triang. (so a triang)
- Q: does every top. mfd admit a triang?
  A: depends on dim. n=0,1: Yes
  n=2 (Rado 1925): every surface has a PL-structure, Yes
  n=3 (Moise 1952): every 3-mfd has a sm structure, yes
  n=4 (Casson): No: Casson invariant shows Freedman E8 mfd is not triangulable
  ↳ Rohklin thm shows E8 mfd has no sm structure
  E8 mfd: 

plumbing diagram. boundary is Poincaré H_*S^3
  Freedman showed any $\mathbb{Z}HS^3$ bounds a compact contractible top. 4-mfd
  n≥5 (Manolescu 2013): No

- <u>Q</u>: does every top mfd admit a PL-structure?
  A. $n = 0,1,2,3$: Yes as before
  $n=4$: Freedman $E8$ has no PL-str. No
  $n \geq 5$ (Kirby-Siebenmann): <u>No</u>, M top mfd, Kirby-Siebenmann invariant
  $\Delta(M) \in H^4(M, \mathbb{Z}_2)$. in $n \geq 5$, $\Delta(M) = 0 \Leftrightarrow M$ admits PL-str,
  in $n=4$, M admits PL-str $\Rightarrow \Delta(M) = 0$
  4 e.g. $\Delta(S^1 \times E8) \neq 0$ so $S^1 \times E8$ is a top mfd w/ no PL str.
  $\Delta(T^{n-4} \times E8) \neq 0$ for $n \geq 5$ generally
- <u>def</u>: M^n top. mfd, $n \geq 5$, diagonal $D \subseteq M \times M$.
  $\nu(D)$ is a $\mathbb{R}^n$-bundle over M called <u>topological tangent bundle</u> TM of M
- <u>def</u>: $TOP(n)$ = homeomorphisms of $\mathbb{R}^n$ fixing 0, $TOP = \lim_{n \to \infty} TOP(n)$ inf dim top gp
  $BTOP$ = classifying space of TOP
  i.e. $\exists\ \Phi : M \to BTOP$ st. TM is pullback:

$$\begin{tikzcd} TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & ETOP \arrow[d] \\ M \arrow[r, "\Phi"] & BTOP \end{tikzcd}$$

$ETOP$ ← weakly contractible space on which TOP acts properly and freely

$PL(n)$ = PL-homeo of $\mathbb{R}^n$ fixing 0, $PL = \lim_{n \to \infty} PL(n) \subset Top$

fibration:

$$\begin{tikzcd} K(\mathbb{Z}_2;3) = TOP/PL \arrow[r] & BPL \arrow[d] \\ & BTOP \end{tikzcd}$$

"does M have a PL triangulation"
can be phrased as "is there a lift"

$$\begin{tikzcd} & BPL \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "\Phi"] & BTOP \end{tikzcd}$$

★ $\Delta(M)$ is obstruction to lifting Φ

- <u>def</u>: assume M^n has triang K (not necessarily PL), for simplicity M orientable, $n \geq 5$

$$c(K) = \sum_{\sigma \in K^{n-4}} [\underbrace{lk(\sigma)}_{HS^3}]\, \sigma \in H_{n-4}(M, \Theta^3_{\mathbb{Z}}) \underset{\text{Poincare duality}}{\cong} H^4(M, \Theta^3_{\mathbb{Z}})$$

s.e.s: $0 \to \ker \mu \to \Theta^3_{\mathbb{Z}} \overset{\mu}{\twoheadrightarrow} \mathbb{Z}_2 \to 0$

induces l.e.s. on cohomology: $\ldots \to H^4(M; \Theta^3_{\mathbb{Z}}) \xrightarrow{\mu} H^4(M; \mathbb{Z}_2) \xrightarrow{\delta} H^5(M; \ker \mu) \to \ldots$

i.e. $\mu(c(K)) =: \Delta(M)$ $\quad \overset{\in}{c(K)} \longmapsto \Delta(M)$

Observe: K combinatorial $\Rightarrow [lk(\sigma)] = 0 \Rightarrow c(K) = 0$

$\mu(c(K)) = \Delta(M) = 0 \Leftrightarrow M$ admits combinatorial triang. (possibly different from K)

les s.e.s M admits triang $\Rightarrow \delta(\Delta(M)) = 0 \in H^5(M, \ker \mu)$

Galewski-Stern, Matumoto showed converse holds.

Also showed s.e.s. splits $\Leftrightarrow \forall n \geq 5, \exists M^n$ w/ $\delta(\Delta(M)) \neq 0$

And Matumoto showed s.e.s doesn't split.

Can use Steenrod squares to give examples of non triangulable top. mfds.

- eg. (Kronheimer): let X be simply conn top 4-mfd w/ intersection form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\Delta(X) \neq 0$ (exists by Freedman). Freedman also implies $\exists$ orientation-reversing homeo $f: X \to -X$. $M := (X \times I)/(x,0) \sim (f(x),0)$ (mapping torus).

exercise: $Sq^1(\Delta(M)) \neq 0$ (so top mfd but not triangulable)

note: all nontriangulable 5-mfds are nonorientable

P^6 = circle bundle over M associated to oriented double cover is also top. but not triangulable

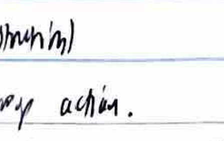
▷ Manolescu's invariant $\beta(Y) \in \mathbb{Z}$ comes from Pin(2)-equivariant Seiberg-Witten Floer homology

- def: quaternions $\mathbb{H} = \{x + yi + zj + wk : x, y, z, w \in \mathbb{R}\} = \mathbb{C} \oplus \mathbb{C}j$

unit quaternions $S(\mathbb{H}) = SU(2) = \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \middle| a^2+b^2+c^2+d^2 = 1 \ (\in \mathbb{R}) \right\}$

$S^1 = \mathbb{C} \cap S(\mathbb{H})$.

$Pin(2) = S^1 \cup S^1 \cdot j \subseteq \mathbb{C} \cup \mathbb{C}j = \mathbb{H}$



$j^2 = -1$

$ij = -ji$

To a 3-mfd Y (with some extra data), one can associate a space I (up to homotopy)

this space admits an action by $Pin(2)$: $SWFH_*^{Pin(2)}(Y) = Pin(2)$-equivariant homology of I

- def: equivariant (co)homology (Borel construction)

goal: homology theory for spaces w/ a group action.

Let X be a space with a top. gp. G acting on it $X \circlearrowleft G$

e.g: $S^2 \circlearrowleft S^1$ by rotation, $S^2/S^1 = I$ (contractible), action is not free (north/south pole fixed)

classifying space BG: EG weakly contractible space on which G acts properly and freely

$$\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$$

[def: weakly contractible = homotopy gps all trivial. for CW complex, weakly contractible $\Leftrightarrow$ contractible]

e.g: $G = \mathbb{Z}$, $EG = \mathbb{R} \to BG = S^1$

$BG = EG/G$. $H^*(BG)$ = group cohomology of G

e.g: $G = \mathbb{Z}_2$, $EG = S^\infty \to \mathbb{R}P^\infty$

e.g: $G = S^1$, $EG = S^\infty \to \mathbb{C}P^\infty$ $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[u]$, $\deg u = 2$

e.g: $G = SU(2)$ exercise: $BSU(2) = \mathbb{H}P^\infty$ and compute $H^*(\mathbb{H}P^\infty; \mathbb{Z})$ $(= \mathbb{Z}[y]$ y deg 4$)$

homotopy quotient: $X \circlearrowleft G$, then $EG \times_G X = EG \times X / G$

G acts on $EG \times X$ via diagonal action. Action of G on $EG \times_G X$ is free

$p: EG \times_G X \to EG/G = BG$: bundle:
$$\begin{tikzcd} X \arrow[r] & EG \times_G X \arrow[d] \\ & BG \end{tikzcd}$$

Borel cohomology or equivariant cohomology of $X \circlearrowleft G$ is $H^*_G(X;R) := H^*(EG \times_G X; R)$

- e.g.: G trivial gp : $H^*_G(X;R) = H^*(X;R)$
- e.g.: X contractible : $H^*_G(X;R) = H^*(BG;R)$
- e.g.: $X \circlearrowleft G$ free : projection $EG \times_G X \to X/G$ homotopy equivalence $\Rightarrow H^*_G(X;R) = H^*(X/G;R)$
- e.g.: $S^2 \circlearrowleft S^1$ by rotation 
$$\begin{tikzcd} S^2 \arrow[r] & S^\infty \times_{S^1} S^1 \arrow[d] \\ & \mathbb{C}P^\infty \end{tikzcd}$$

spectral sequence (↑ cohomology of S^2 , → cohomology of $\mathbb{C}P^\infty$)

$$\begin{array}{ccccccc} H^0(\mathbb{C}P^\infty) & 0 & H^2(\mathbb{C}P^\infty) & 0 & H^4(\mathbb{C}P^\infty) & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ H^0(\mathbb{C}P^\infty) & 0 & H^2(\mathbb{C}P^\infty) & 0 & H^4(\mathbb{C}P^\infty) & \cdots \end{array}$$

d_{2+i} 1+i down
2+i right

- ★ $SWFH_*^{Pin(2)}(Y)$ is a module over $H^*(BPin(2))$
- Q: What is $H^*(BPin(2))$?

$Pin(2) = S^1 \cup S^1 j \subset SU(2) \subset \mathbb{H}$ fibration:
$$\begin{tikzcd} Pin(2) \arrow[r] & SU(2) \arrow[d] \\ & \mathbb{R}P^2 \end{tikzcd}$$

i.e.
$$\begin{tikzcd} S^1 \cup S^1 j \arrow[r] & \{x + yi + zj + wk : x^2+y^2+z^2+w^2 = 1 \text{ (or } k)\} \arrow[d, "p"] \\ & \mathbb{R}P^2 \end{tikzcd}$$

exercise: p is composition of Hopf fibration map and antipodal map on S^2

fibration:
$$\begin{tikzcd} \mathbb{R}P^2 \arrow[r] & BPin(2) \arrow[d] \\ & BSU(2) = \mathbb{H}P^\infty \end{tikzcd}$$

spectral sequence ($\mathbb{F} = \mathbb{Z}_2$ coeffs) $H^*(\mathbb{H}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[y]$ deg $y = 4$

$$\begin{array}{c|cccccccccc} & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} & \cdots \\ \uparrow & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} & \cdots \\ Q \\ V \to & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} & \cdots \end{array}$$

no room for higher differentials

$H^*(BPin(2), \mathbb{F}) = \mathbb{F}[Q,V]/(Q^3)$ deg $Q = 1$, deg $V = 4$

★ equivariant (co)-homology is a module over $H^*(BG;R)$

$\boxed{\mathbb{F} = \mathbb{Z}_2}$

e.g. S^1-equivariant homology: $S^1 \to S^\infty$ $\downarrow$ $\mathbb{C}P^\infty = BS^1$ $H^*(\mathbb{C}P^\infty; \mathbb{F}) = \mathbb{F}[U]$ deg $U = 2$

so $H_*^{S^1}(X; \mathbb{F})$ module over $\mathbb{F}[U]$

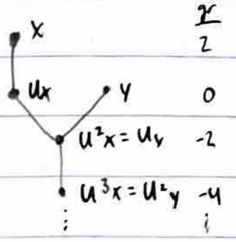
$\mathbb{F}[U]$ is a PID: any f.g. module M over $\mathbb{F}[U]$ (any PID) is (non-canonically) $\cong$ to $\bigoplus_{j=1}^{N} \mathbb{F}[U] \oplus \bigoplus_{i=1}^{m} \mathbb{F}[U]/(p_i)$. Moreover if M is graded then each poly. p_i must be homogeneously graded, i.e. $p_i = U^{m_i}$ for some m_i.

Hence: $M = \bigoplus_{j=1}^{N} \mathbb{F}_{d_j}[U] \oplus \bigoplus_{i=1}^{M} \mathbb{F}_{c_i}[U]/U^{m_i}$ (where $\mathbb{F}_d[U] = \mathbb{F}[U]$ where $gr\, 1 = d$)

* convention: to line up w/ Heegaard Floer, from now, deg $U = -2$

Sps $N = 1$. then define $d(M) = \max\{gr(x) : x \in M,\ U^k x \neq 0\ \forall k > 0\}$

e.g. $M =$



$d(M) = 2$

$M \cong \mathbb{F}[U] \oplus \mathbb{F}[U]/(U) \cong \mathbb{F}[U] \oplus \mathbb{F}$

$\cong \mathbb{F}[U]\langle x \rangle \oplus \mathbb{F}\langle Ux + y \rangle$

for $k \gg 0$, $U^k M$ is 1-dim

- eg. Pin(2)-equivariant homology: $H^*(BPin(2); \mathbb{F}) = \mathbb{F}[Q, V]/Q^3$

convention: deg $Q = -1$ deg $V = -4$.

$SWFH_*^{Pin(2)}(Y)$ is a module over $\mathbb{F}[Q,V]/Q^3$ (note not a PID, e.g. $\langle Q, V\rangle$)

Manolescu proved for $N \gg 0$, $V^N \cdot SWFH_*^{Pin(2)}(Y)$ is 3-dim: x_1, x_2, x_3

and $Qx_3 = x_2$, $Q^2 x_3 = Qx_2 = x_1$ "picks out x_1"

defn: $A(Y) = \max\{gr(x) : x \in SWFH_*^{Pin(2)}(Y)$, for $N \gg 0$, $V^N \cdot x \neq 0$ and $V^N \cdot x \in \operatorname{Im} Q^2\}$

$B(Y) = \max\{\ldots\ V^N x \neq 0,\ QV^N \cdot x \neq 0,\ Q^2 V^N \cdot x = 0\}$ "x_2"

$C(Y) = \max\{\ldots\ V^N \cdot x = 0,\ Q^2 V^N \cdot x \neq 0\}$ "x_3"

Renormalize: $\alpha = A/2$, $\beta = B-1/2$, $\gamma = C-2/2$

- thm (Manolescu): 1) α, β, γ are invariants of homology cobordism.

2) β mod 2 = Rokhlin invariant   3) $\beta(-Y) = -\beta(Y)$

$SWFH_*^{Pin(2)}(Y)$ is closely related to involutive H-F homology, a refinement of H-F theory.

Heegaard Floer Homology ($\mathbb{F} = \mathbb{Z}_2$)

- **Overview**:

Heegaard diagram $\mathcal{H}$ for 3-mfd Y $\rightsquigarrow$ chain complex $CF^-(\mathcal{H})$ free, f.g. graded chain complex over $\mathbb{F}[U]$ $\rightsquigarrow$ $HF^-(Y) = H_*(CF^-(Y))$ f.g. graded module over $F[U]$ (deg $U = -2$)

remark: $HF^-(Y) \cong SWFH^{S^1}_*(Y)$

note: $HF^-(Y) = \bigoplus_{s \in spin^c(Y)} HF^-(Y,s)$

remark: $Spin^c(Y) \overset{1:1}{\leftrightarrow} H_1(Y;\mathbb{Z}) \cong H^2(Y;\mathbb{Z})$

(Osváth - Szabo): $\underbrace{Y \text{ a } QHS^3}_{H_1(Y;\mathbb{Z}) \text{ is finite}}$, $HF^-(Y,s) \cong \mathbb{F}[U] \oplus \bigoplus_{i=1}^{m} \mathbb{F}[U_i]/U^{n_i}$ $\forall s \in Spin^c(Y)$

A cobordism $W: Y_0 \to Y_1$ induces a module homomorphism $F_W: HF^-(Y_0) \to HF^-(Y_1)$

- **Other Flavors**:

- s.e.s: $0 \to \mathbb{F}[U] \xrightarrow{\cdot U} \mathbb{F}[U] \to \mathbb{F} \to 0$

$0 \to CF^-(\mathcal{H}) \xrightarrow{\cdot U} CF^-(\mathcal{H}) \to \widehat{CF}(\mathcal{H}) \to 0$

$\widehat{CF}(\mathcal{H})$ obtained from $CF^-(\mathcal{H})$ by setting $U=0$

$\widehat{HF}(Y)$ $:= H_*(\widehat{CF}(\mathcal{H}))$ - weaker than HF^- but sometimes easier to work with

eg: $CF^-(\mathcal{H}) = \langle x, y, z \rangle_{\mathbb{F}[U]}$ $\partial x = 0$, $\partial y = Uz$, $\partial z = 0$

$\ker \partial = \langle x, z \rangle$, $\operatorname{Im} \partial = \langle Uz \rangle$

$H_*(CF^-(\mathcal{H})) \cong \mathbb{F}[U]\langle x \rangle \oplus \mathbb{F}\langle z \rangle$

[set $U=0$ then take homology]

but $\widehat{CF}^-(\mathcal{H}) = \langle x, y, z \rangle_{\mathbb{F}[U]}$ $\partial x = \partial y = \partial z = 0$

$\ker \partial = \langle x, y, z \rangle$, $\operatorname{Im} \partial = 0$, so $H_*(\widehat{CF}(\mathcal{H})) = \mathbb{F}^3$

exercise: show $\widehat{HF}(Y)$ is determined by $HF^-(Y)$

- s.e.s: $0 \to \mathbb{F}[U] \hookrightarrow \mathbb{F}[U, U^{-1}] \to \mathbb{F}[U, U^{-1}]/\mathbb{F}[U] \to 0$

$$0 \to CF^-(\mathcal{H}) \hookrightarrow \underbrace{CF^-(\mathcal{H}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]}_{CF^\infty(\mathcal{H})} \to CF^+(\mathcal{H}) \to 0$$

$HF^+(Y)$ $:= H_*(CF^+(\mathcal{H}))$

$HF^\infty(Y)$ $:= H_*(CF^\infty(\mathcal{H}))$ (turns out to be boring)

if Y QHS^3, $HF^\infty(Y,s) \cong \mathbb{F}[U, U^{-1}]$ one spin^c

exercise: If Y QHS^3, show $HF^+(Y)$ is determined by $HF^-(Y)$ and vice versa

- **Notation**: Write $CF^\circ(Y)$, $HF^\circ(Y)$ where $\circ = \wedge, +, -, \infty$ (circle)

- knots in 3-mfds:

A nullhomologous knot in Y induces a filtration on $CF^\circ(\mathcal{H})$

↳ if $Y = S^3$ or $\mathbb{Z}HS^3$, all knots are nullhomologous

↳ filtration convention: $\ldots \subseteq F_{i-1} \subseteq F_i \subseteq F_{i+1} \subseteq \ldots$

the associated graded complex is $gCFK^\circ(Y,K) = \bigoplus_i F_i/F_{i-1}$

knot Floer homology: $HFK^\circ(Y,K) = H_*\left(\bigoplus_i F_i/F_{i-1}\right)$

- bigraded: homological (Maslov) grading (∂ decreases by 1) (m)
  Alexander grading (coming from filtration) (s)

$\widehat{HFK}(Y,K)$ with $\mathbb{F} = \mathbb{Z}_2$ coeff is bigraded vector space.

thm (Ozsváth - Szabó): $\widehat{HFK}(S^3,K)$ categorifies the Alexander polynomial; i.e.

$\Delta_K(t) = \sum_{m,s} (-1)^m t^s \dim(\widehat{HFK}_m(K,s))$ Alexander grading (like quantum grading)

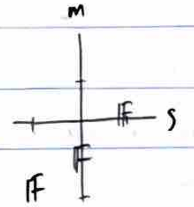
homological grading (Maslov)

e.g.: $K = T_{2,3}$

exercise: $\deg \Delta_K(t) \leq g(K)$

symmetrized

$\widehat{HFK}(K)$:



↙ symmetrized, deg 1 (t^1)

$t^{-1} - 1 + t = \Delta_K(t)$

thm (Ozsváth - Szabó): $\widehat{HFK}$ detects genus (tells you what genus is):

$g(K) = \max\{s : \widehat{HFK}(K,s) \neq 0\}$ (notation: drop S^3 since knots usually live here)

def: a knot $K \subseteq S^3$ is fibered if $S^3 \setminus K$ is a fiber bundle over S^1

exercise: if K is fibered then $\Delta_K(t)$ is monic

rightmost grading

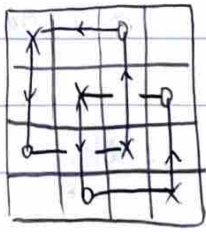
thm (Ghiggini, Ni): $\widehat{HFK}$ detects fiberedness, i.e., K is fibered $\Leftrightarrow$ $\widehat{HFK}(K, g(K)) \cong \mathbb{F}$

- Grid homology ($K \subseteq S^3$)

def: A planar grid diagram $\mathbb{G}$ is a $n \times n$ grid st. n squares are marked X's and n are marked O's. st.

1) each col has exactly 1 X, 1 O
2) each row has exactly 1 X, 1 O
3) no square has an X and O

e.g.

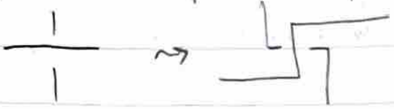


to get an oriented link, connect X → O in each col, in each row, connect O → X st. vertical strands are over

Hopf link

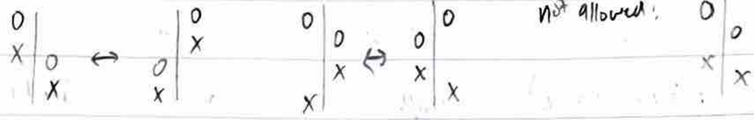
Q: Can every link be represented by a grid diagram?

A: Yes



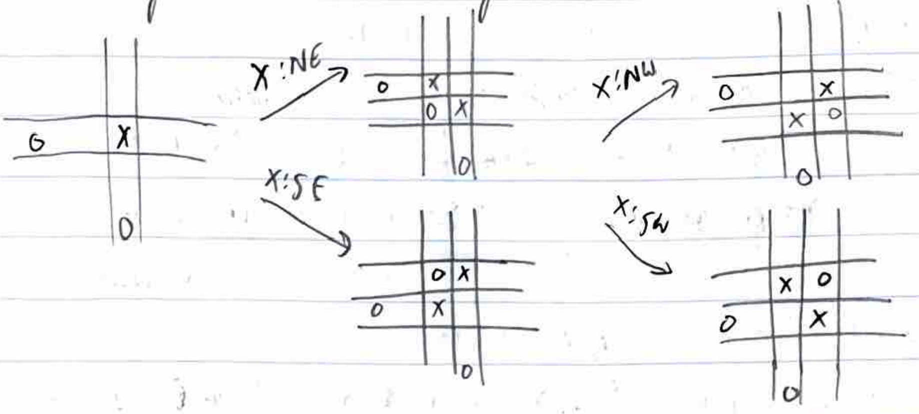
grid moves:

- column commutation



- row commutation
- (de)stabilization

$n \times n$ grid $\overset{\text{stab}}{\rightsquigarrow}$ $(n+1)\times(n+1)$ grid



exercise: check these moves preserve isotopy class of link

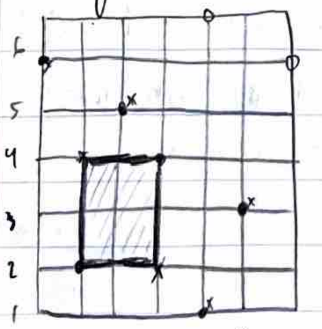
thm (Cromwell): Grid diagrams represent same link iff related by finite seq. of grid moves

toroidal grid diagrams

cyclic perm. of the rows/cols don't change the knot

goal: bigraded chain complex whose homology is knot inv. and graded Euler char is $\Delta(t)$

def. grid states $S(G)$ = bijections between vertical grid horizontal circles = S_n



↳ generators for chain complex

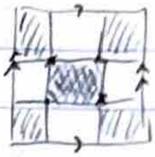
eg: $y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 5 & 4 & 1 & 3 \end{pmatrix}$ (o)

$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix}$ (x)

Sps $y, z \in S(G)$ st. y, z agree in exactly $n-2$ pts. Consider the 2 pts in y and 2 in z where they disagree. (differ by a transposition) → gives a rectangle r in the grid

r goes from y to z - (NW, SE, terminal corners)
(NE, SW corners, initial corners)

$\mathrm{Rect}(y,z) = \{$rectangles from y to z$\}$



note 1) $|\mathrm{Rect}(y,z)| = \begin{cases} 2 & \text{if } y, z \text{ differ in exactly 2 pts} \\ 0 & \text{else} \end{cases}$

2) $r \in \mathrm{Rect}(y,z)$, then $y \cap \mathrm{Int}(r) = z \cap \mathrm{Int}(r)$

def: A rectangle $r \in \mathrm{Rect}(y,z)$ is empty if $y \cap \mathrm{Int}(r) = z \cap \mathrm{Int}(r) = \emptyset$.

$\mathrm{Rect}^{\circ}(y,z) = \{$empty rectangles $r \in \mathrm{Rect}(y,z)\}$

bigrading: $p = (p_1, p_2)$, $q = (q_1, q_2)$. define $p < q$ if $p_1 < q_1$ and $p_2 < q_2$

$\cdot q$ $\cdot p$

$p \cdot$ $p < q$ $\cdot q$ $p \not< q$, $q \not< p$.

def: let P, Q be finite collection of pts in $\mathbb{R}^2$. define

$I(P,Q) = |\{(p,q) : p \in P, q \in Q, p < q\}|$, $J(P,Q) = \frac{1}{2}(I(P,Q) + I(Q,P))$.

$$\underbrace{\mathbb{O} = \text{set of O's in grid diagram}, \ \mathbb{X} = \text{set of X's}}_{\text{half integer coords}}, \ \underbrace{y \in S(G)}_{\text{integer coords}}.$$

fundamental domain: $[0,n) \times [0,n)$

def: $M_{\mathbb{O}}(y) = J(y,y) - 2J(y,\mathbb{O}) + J(\mathbb{O},\mathbb{O}) + 1 = J(y-\mathbb{O}, y-\mathbb{O}) + 1$

$M_{\mathbb{X}}(y) = J(y-\mathbb{X}, y-\mathbb{X}) + 1$

def: Maslov grading: $M(y) = M_{\mathbb{O}}(y)$

Alexander grading: $A(y) = \frac{1}{2}(M_{\mathbb{O}}(y) - M_{\mathbb{X}}(y)) - (n-1)/2$

Prop: $M: S(G) \to \mathbb{Z}$, $A: S(G) \to \mathbb{Z}$ are well def. Moreover, M characterized by:

1) let y^{NWO} be the grid state consisting of the upper left corners of the O squares. then $M(y^{NWO}) = 0$

2) If $|\mathrm{Rect}(y,z)| \neq 0$, then $M(y) - M(z) = 1 - 2|\{r \cap \mathbb{O}\}| + 2|\{y \cap \mathrm{Int}(r)\}|$

↳ transpositions generate S_n so (2) tells you how to get to any other generator $S(G)$.

Up to an additive const, A is characterized by $r \in \mathrm{Rect}(y,z)$,

$A(y) - A(z) = |\{r \cap \mathbb{X}\}| - |\{r \cap \mathbb{O}\}|$

fully blocked grid chain complex

$\widetilde{GC}(\mathbb{G})$ = bigraded chain complex over $\mathbb{F} = \mathbb{Z}_2$, generated by $S(\mathbb{G})$

$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(y) = \sum_{z \in S(\mathbb{G})} \#_{\text{mod } 2} \underbrace{\{ r \in Rect^\circ(y,z) \mid r \cap \mathbb{X} = r \cap \mathbb{O} = \emptyset \}}_{Rect^\circ_{\mathbb{O},\mathbb{X}}(y,z)} \cdot z$

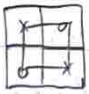
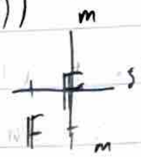
↑ differential

(either 0 or 1, 2 m

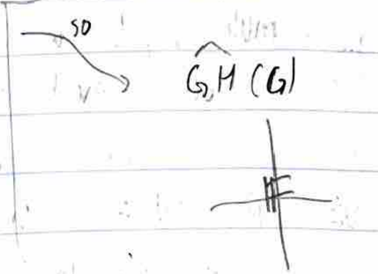
$Rect^\circ_{\mathbb{O},\mathbb{X}}(y,z)$ - "fully empty rects from $y \to z$"

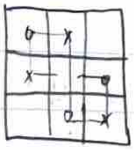
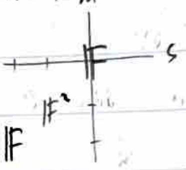
exercise: $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ lowers Maslov grading by 1 and preserves Alexander grading

defn: $\widetilde{GH}(\mathbb{G}) = H_*(\widetilde{GC}(\mathbb{G}))$

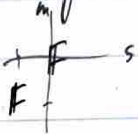
exercise: 1)  has $\widetilde{GH}(\mathbb{G})$: 

($\partial = 0$, no fully empty rects)



2) $\mathbb{G}$ =  has $\widetilde{GH}(\mathbb{G})$: 

* $\widetilde{GH}(\mathbb{G})$ is not a knot invariant

Thm: let $\mathbb{G}$ be a $n \times n$ toroidal grid diagram for K. Let W be the 2-dim vector space: 

. Then ∃ bigraded vector space $\widehat{GH}(\mathbb{G}) = \widehat{HFK}(K)$ s.t.

1) $\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes n-1}$ → "untensor W" to recover $\widehat{GH}$

2) $\widehat{GH}(\mathbb{G})$ is a knot invariant

Prop: $\partial^2_{\mathbb{X},\mathbb{O}} = 0$

Pf: Fix $y \in S(\mathbb{G})$. Then $\partial^2_{\mathbb{X},\mathbb{O}}(y) = \partial_{\mathbb{X},\mathbb{O}}\left(\sum_{z \in S(\mathbb{G})} \# Rect^\circ_{\mathbb{X},\mathbb{O}}(y,z) \cdot z \right)$

$= \sum_{z \in S(\mathbb{G})} \# Rect^\circ_{\mathbb{X},\mathbb{O}}(y,z) \sum_{w \in S(\mathbb{G})} \# Rect^\circ_{\mathbb{X},\mathbb{O}}(z,w) \cdot w$

Sps $r_1 \in Rect^\circ_{\mathbb{X},\mathbb{O}}(y,z)$, $r_2 \in Rect^\circ_{\mathbb{X},\mathbb{O}}(z,w)$..

Case1: Corners of r_1, r_2 are all distinct.

Then $\exists!\, z' \in S(G)$ and rects $r_1' \in Rect^\circ_{\mathbb{X},\mathbb{O}}(y,z')$, $r_2' \in Rect^\circ_{\mathbb{X},\mathbb{O}}(z',w)$

s.t. (r_1 and r_2) and (r_1' and r_2') have the same support, so cancel in ∂^2 (mod 2)

Case2: Rectangles r_1, r_2 share a corner

Case3: r_1, r_2 share 2 corners (must overlap along 2 edges (torus))

this can't happen b/c a row/col would have no X's nor O's (r₁ and r₂ are totally empty) 

□

Idea behind $\widehat{GH}$ is a knot invariant:

Show commutation doesn't change homology, (de)stabilization (removes) tensors on W

def: $\widehat{HFK}(K) := \widehat{GH}(G)$,

Other flavors:

$GC^-(G)$ = chain complex generated by $S(G)$ over $\mathbb{F}[U_1, \dots, U_n]$ (G n×n grid)

$\partial^-_{\mathbb{X}}(y) = \sum_{z \in S(G)} \sum_{r \in \mathrm{Rect}^\circ_{\mathbb{X}}(y,z)} U_1^{n_{O_1}(r)} \cdots U_n^{n_{O_n}(r)} \cdot z$, $\mathrm{Rect}^\circ_{\mathbb{X}}(y,z)$ = rects $y \to z$ with no X's,

$n_{O_i}(r)$ = # each O_i appears in r (label O's $O_1, \dots, O_n$)

Prop: mult. by U_i and U_j are chain homotopic: i.e. $\exists\, H: GC^-(G) \to GC^-(G)$

s.t. $U_i + U_j = \partial^-_{\mathbb{X}} H + H \partial^-_{\mathbb{X}}$.

So $H_*(GC^-(G))$ is a module over $\mathbb{F}[U]$ (all the U_i's collapse to one var in homology)

def: $HFK^-(K) := H_*(GC^-(G))$

exercise: $H_*(GC^-(G)/U_1=0) \otimes W^{n-1} \cong H_*(\widetilde{GC}(G))$

so $H_*(GC^-(G)/U_1=0) = \widehat{HFK}(K)$

Concordance Invariants:

Allowing rectangles to contain $\mathbb{X}$'s, Alexander grading becomes a filtration:

$A(\partial y) \leq \partial(Ay)$, so $\emptyset = \dots \subseteq F_{s-1} \subseteq F_s \subseteq F_{s+1} \subseteq \dots = GC(G)/U_1=0$ ← any filter

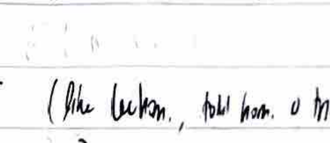
↳ like Lee homology: extra differential means Alex. grading is not an invariant anymore but gives a filtration.

Moreover, total homology w.r.t. this differential is $\mathbb{F}$ (like Lee hom., total hom. is trivial)

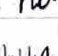
def: $\tau(K) = \min\{s \mid F_s \hookrightarrow GC(G)/_{U_1=0}$ surjective on $H_*\}$

Heegaard Diagrams

- **def**: a handlebody of genus g is a closed regular nbhd of a wedge of 3 circles in $\mathbb{R}^3$
Equivalently, $B^3 \cup$ (g 3dim 1-handles)



- **def**: A Heegaard splitting of a closed oriented 3-mfd
Y is a decomp $Y = H_1 \cup_\varphi H_2$, H_1, H_2 handlebodies, φ orientation reversing homeo.
- **def**: the genus of a splitting is the genus of ∂H_1 (or ∂H_2). $\Sigma = \partial H_1 = -\partial H_2$
is the Heegaard surface
- **thm**: Every 3 mfd admits a Heegaard splitting.
Pf: Every 3-mfd admits triangulation. H_1 = nbhd of 1-skeleton, H_2 = nbhd of dual of 1-skeleton



□

- def: let H be a handlebody of genus g. A set of attaching curves for H is a set $\{\gamma_1, \dots, \gamma_g\}$ of simple closed curves in ∂H st
  1) curves pairwise disjoint 2) $\Sigma - \{\gamma_1, \dots \gamma_g\}$ is connected
  3) each γ_i bounds a disk in H

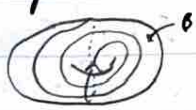
e.g.

attaching curves tell you how to fill in surface: glue in thickened disks along γ_i's, can see S^2 in ∂, unique way to glue in 3-ball

- def: a Heegaard diagram compatible w/ a H-splitting $Y = H_1 \cup_{\Sigma} H_2$ is $\mathcal{H} = (\Sigma, \alpha, \beta)$ where 1) Σ closed, orientable, genus g surface
  2) $\alpha = \{\alpha_1, \dots, \alpha_g\}$ attaching curves for H_1 3) $\beta = \{\dots\}$ … for H_2

e.g.

S^3



$H_1(3\text{mfd}) = \mathbb{Z}_2$ so $\mathbb{RP}^3$

- Given Heegaard diagram (Σ, α, β), build 3-mfd by
  1) thicken Σ to $\Sigma \times I$
  2) along $\Sigma \times \{0\}$, attach thickened disks to $\alpha \times \{0\}$ 3) .. $\Sigma \times \{1\}$, … $\beta \times \{1\}$
  exercise: ∂ of this 3-mfd is $S^2 \sqcup S^2$
  4) fill in S^2's w/ B^3 (unique way b/c unique orientation preserving $B^3 \circlearrowleft$ is id)

e.g.

$\Sigma \times \{0\}$:

$\Sigma \times \{1\}$:

- thm: two Heegaard diagrams describe same 3mfd $\Leftrightarrow$ they are related by finite sequence of 1) isotopy 2) handleslides 3) (de-)stabilization

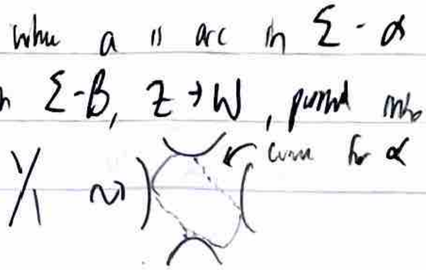
3: [connect sum w/ (T^2, α, β) where α, β s.c.c. intersect transversally once)

for technical reasons, need a basept $w \in \Sigma$, and isotopies / handleslides can't cross w.

- def: a doubly pointed Heegaard diagram for knot $K \subset Y$ is $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$
  1) (Σ, α, β) Heegaard diagram for Y
  2) K is the union of 2 arcs a and b where a is arc in $\Sigma - \alpha$ connecting w to z pushed into H_1, b is in $\Sigma - \beta$, $z \to w$, pushed into H_2.

see Week 10 pictures

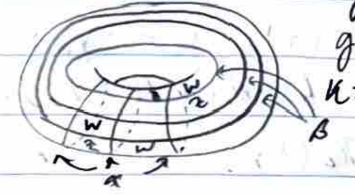
- from knot diagram, obtain doubly pointed H-diagram for K by



- thm: two doubly pointed Heegaard diagrams represent same knot iff related by finite seq of doubly pointed isotopies (don't cross w/z), handleslides, (de)stabilization
- generally: let (Σ, α, β) consist of 1) genus g surface Σ
  2) $g+k$ disjoint s.c.c. $\alpha_1, \ldots, \alpha_{g+k}$ spanning half-dim subspace of $H_1(\Sigma, \mathbb{Z})$
  3) ... $\beta_1, \ldots, \beta_{g+k}$ ... ...

build 3mfd: 1) thicken Σ 2) attach thickened disks to $\alpha_i \times \{0\}$, $\beta_i \times \{1\}$
3) attach $2(k+1)$ B^3's along boundary components

e.g.: S^3



$g=1$ [like toroidal grid diagram]
$k=2$

add basepts $w_1, \ldots, w_{k+1}$, $z_1, \ldots, z_{k+1}$ st.
1) each connected comp of $\Sigma - \alpha_1 - \ldots - \alpha_{g+k}$ has exactly one w_i, one z_i
2) ... $\Sigma - \beta_1 - \ldots - \beta_{g+k}$ ... ...

this specifies a knot as before: connect $w \to z$ in $\Sigma - \alpha_1 - \ldots \alpha_{g+k}$, $z \to w$ in $\Sigma - \beta_1 \ldots \beta$

- generators: $(\Sigma, \alpha, \beta, w)$, g-tuples of intersection pts between α-circles and β-circles st. each α-circle (resp β-circle) is used exactly once
  ↳ see analogy to $S(\mathbb{G})$ grid states

[see example and exercise]

$\mathrm{Sym}^g(\Sigma) = (\Sigma \times_{g} \ldots \times \Sigma)/S_g$ ← symmetric grp on g elts = unordered g-tuple of pts in Σ
↳ rk: action of S_g on $\Sigma \times \ldots \times \Sigma$ is usually not free
↳ $\mathrm{Sym}^g(\Sigma)$ is a sm mfd though!

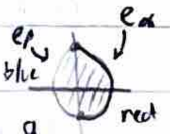
exercises: see lecture notes on HF

half dim subspaces $\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g$, $\mathbb{T}_\beta = \ldots \subset \mathrm{Sym}^g(\Sigma)$
$\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma)$ and H-F generators are exactly $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

also $V_w := w \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$ unordered g-tuples of pts in Σ st. at least one pt is w

H-F differential: $\widehat{CF}(\mathcal{H}) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_{\mathbb{F}}$ $(\mathbb{F} = \mathbb{Z}_2)$

$\partial : \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$ "counts holomorphic disks"

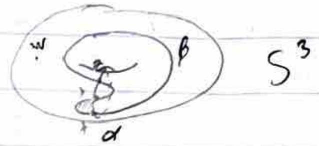


consider $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\varphi : D \to \mathrm{Sym}^g(\Sigma)$ (D complex unit disk) st.
1) $\varphi(-i) = x$ 2) $\varphi(i) = y$ 3) $\varphi(e_\alpha) \subset \mathbb{T}_\alpha$ 4) $\varphi(e_\beta) \subset \mathbb{T}_\beta$

D is a Whitney disk from x to y. $\pi_2(x, y) := \{$homotopy classes of Whitney disks $x \to y\}$

we can picture im(φ) via "shadow" in Σ (for $g=1$, $\mathrm{Sym}^g(\Sigma)=\Sigma$)

e.g.:



(each α, β bound a "disk", gluing creates a S^2

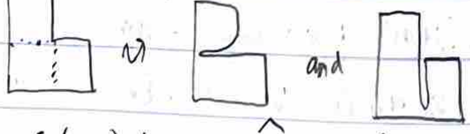
there are like rectangles in grid homology

- Observe: given $x=\{x_1,\dots,x_g\}$, $y=\{y_1,\dots,y_g\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, define $\varepsilon(x,y) \in H_1(Y;\mathbb{Z})$ as follows: choose arcs $a \in \alpha$'s, $b \in \beta$'s st. $\partial a = y_1 + \dots + y_g - x_1 - \dots - x_g$, $\partial b = x_1 + \dots + x_g - y_1 - \dots - y_g$. Then $a+b$ is a 1-cycle in Σ and $\varepsilon(x,y) = [a+b] \in H_1(Y;\mathbb{Z})$ is well def.
- exercise: If $\varepsilon(x,y) \neq 0 \in H_1(Y;\mathbb{Z})$, then $\pi_2(x,y)=\emptyset$ (no Whitney disks)
- Technical details: choose a complex str on Σ, induces complex str on $\mathrm{Sym}^g(\Sigma)$. given $\varphi \in \pi_2(x,y)$, let $\mathcal{M}(\varphi)$ :: moduli space of holomorphic reps of φ. $\mu(\varphi)$ = expected dim of $\mathcal{M}(\varphi)$. there is $\mathbb{R}$-action on $\mathcal{M}(\varphi)$ coming from automorphisms of D fixing $\pm i$. $\hat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi)/\mathbb{R}$. $n_w(\varphi)$ = alg. intersection # b/w $\varphi(D)$, V_w
- HF differential: $\hat{\partial}: \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$. $\hat{\partial}x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{*} \# \hat{\mathcal{M}}(\varphi)\, y$ $\left[*: \begin{array}{l} \varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0 \end{array} \right]$

rk: 1) $\widehat{CF}(\mathcal{H})$ is relatively graded: $gr(x) - gr(y) = \mu(\varphi) - 2n_w(\varphi)$ ($\varphi \in \pi_2(x,y)$), can lift to an absolute grading via normalization $\widehat{HF}(S^3) = \mathbb{F}_{(0)}$ (gr 0) everything is cobordant to S^3, can determine grading of cobordism

2) $\hat{\partial}^2 = 0$ [Idea: 0-dim moduli space $\hat{\mathcal{M}}(\varphi)$ appear as boundary of compact 1-dim moduli space, boundary of a compact 1 dim mfd is even # of pts •—•)

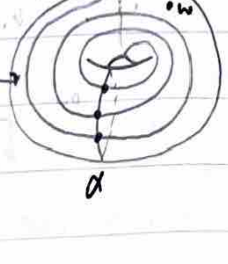
same idea as



3) Since $\pi_2(x,y) = \emptyset$ if $\varepsilon(x,y) \neq 0$, $(\widehat{CF}(\mathcal{H}), \hat{\partial})$ splits as a direct sum (splitting along spin^c structures)

- HF homology: $\widehat{HF}(\mathcal{H}) := H_*(\widehat{CF}(\mathcal{H}))$

e.g.:



$L(3,1)$

no Whitney disks so no differential

$\widehat{HF}(L(3,1)) = \mathbb{F}^3$

$(\widehat{HF}(L(p,q)) = \mathbb{F}^p)$

- exercise: Y $\mathbb{Q}HS^3$, then $\dim(\widehat{HF}(Y)) \geq |H_1(Y;\mathbb{Z})|$ (compute H_1 from Heegaard diagram)

- **def**: Y $\mathbb{Q}HS^3$ and $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. then Y is called an **L-space**.
- **Minus Flavor**: $CF^-(\mathcal{H}) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_{\mathbb{F}[U]}$ $\deg U = -2$

$$\partial^- x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\varphi \in \pi_2(x,y),\, \mu(\varphi)=1} \# \widehat{\mathcal{M}}(\varphi)\, U^{n_w(\varphi)} y$$

$HF^-(\mathcal{H}) := H_*(CF^-(\mathcal{H}))$

[to show $\widehat{HF}$, HF^- are 3-mfd invariants, need to show that Heegaard moves (isotopy, handleslides, stab.) induce chain homotopy equivalences and is indep from choice of complex str.]

rk: $\widehat{CF}(\mathcal{H})$ obtained from $CF^-(\mathcal{H})$ by setting $U=0$.

Recall: s.e.s $0 \to CF^-(Y) \to CF^\infty(Y) = CF^-(Y) \otimes_{\mathbb{F}[U]} \mathbb{F}[U,U^{-1}] \to CF^+(Y) \to 0$

- **thm** (Ozsváth-Szabó): Y $\mathbb{Q}HS^3$ then $HF^\infty(Y,\mathfrak{s}) = \mathbb{F}[U,U^{-1}]$ ($\mathfrak{s}$ spin^c str)

Consequence: Y $\mathbb{Q}HS^3$ then $HF^-(Y,\mathfrak{s}) \cong \mathbb{F}_{(d)}[U] \oplus \bigoplus_{i=1} \mathbb{F}_{(c_i)}[U]/U^{n_i}$ $[\mathbb{F}_{(d)} \leftarrow$ grading of $1 \in \mathbb{F}[U]]$

- **def**: $d(Y,\mathfrak{s}) = \max\{gr(x) : x \in HF^-(Y,\mathfrak{s}),\ U^n x \neq 0,\ \forall n > 0\}$
  - ↳ some slightly different choices of grading normalization in literature
- **eg**:

$HF^-(S^3) = \mathbb{F}_{(0)}[U]$ (other norm. choice is $\mathbb{F}_{(-2)}[U]$)

$d(S^3) = 0$

eg: $HF^-(S^3_{+1}(\overset{\text{trefoil}}{T_{2,3}})) = \mathbb{F}_{(-2)}[U]$, $d(S^3_{+1}(T_{2,3})) = -2$ | $+, \infty$ ↘

- a cobordism $W: Y_0 \to Y_1$ induces a $\mathbb{F}[U]$-module homomorphism $HF^-(Y_0) \to HF^-(Y_1)$
- a compact 4-mfd W is <u>negative definite</u> if its intersection form is negative definite

for simplicity, focus on Y $\mathbb{Z}HS^3$ (can be diagonalized to $\begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}$)

- **thm** (Ozsváth-Szabó): If $W: Y_0 \to Y_1$ is a neg. def. cobordism, then W induces an $\cong$ on HF^∞

must send free part to free part: $HF^-(Y,\mathfrak{s}) = \underbrace{\mathbb{F}_{(d)}[U]}_{\text{"}HF^\infty\text{ part"}} \oplus \bigoplus_{i=1}^{n} \mathbb{F}_{(c_i)}[U]/U^{n_i}$

$\mathbb{F}_{(d_0)}[U] \to \mathbb{F}_{(d_1)}[U] : 1 \mapsto U^n$ (for some n)

Consequence: $d(Y_0) \leq d(Y_1)$. In particular, if Y_0, Y_1 homology cob, then $d(Y_0) = d(Y_1)$

Connected Sum

$\mathcal{H}_1, \mathcal{H}_2$ Heegaard diagrams for Y_1, Y_2. Then $\mathcal{H}_1 \# \mathcal{H}_2$ is a H.D. for $Y_1 \# Y_2$

Prop: $\widehat{CF}(\mathcal{H}_1 \# \mathcal{H}_2) \cong \widehat{CF}(\mathcal{H}_1) \otimes \widehat{CF}(\mathcal{H}_2)$ (take # near basept, consider generators / differential)

↳ analogous result holds for CF^-, harder to prove (on chain level, $\otimes_{\mathbb{F}[U]}$)

↳ $\widehat{HF}(Y_1 \# Y_2) = \widehat{HF}(Y_1) \otimes_{\mathbb{F}} \widehat{HF}(Y_2)$

↳ $HF^-(Y_1 \# Y_2) = H_*(CF^-(Y_1) \otimes_{\mathbb{F}[U]} CF^-(Y_2))$ ↙ $\mathbb{F}[U]$ not a field so $\otimes$ and homology don't commute

- **exercise**: $d(Y_1 \# Y_2) = d(Y_1) + d(Y_2)$
- **Fact**: Y $\mathbb{Z}HS^3$, $d(Y)$ even

- e.g: $d(\Sigma(2,3,5)) = -2$
$\Rightarrow d : \Theta^3_{\mathbb{Z}} \to 2\mathbb{Z}$ is a surjective homomorphism
- Knot Floer Homology.

doubly pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$

$\widehat{CFK}(S^3, K)$ filtered chain complex.

$\widehat{CFK}(\mathcal{H}) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_{\mathbb{F}}$ ← generators are same

relative Alexander grading on generators: $A(x) - A(y) = n_z(\phi) - n_w(\phi)$ $(\phi \in \pi_2(x,y))$

$\hat{\partial} x$ differential is same as $\widehat{CF}(\mathcal{H})$

Alexander grading on generators induces Alexander filtration on $\widehat{CFK}(\mathcal{H})$:

$A(\sum x_i) := \max\{A(x_i)\}$ (max A. grading in sum)

$A(x) \geq A(\hat{\partial} x)$ (* why?)

filtration: $\mathcal{F}_s(\widehat{CFK}(\mathcal{H})) = \langle x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta : A(x) \leq s \rangle_{\mathbb{F}}$

$\dots \subseteq \mathcal{F}_{s-1} \subseteq \mathcal{F}_s \subseteq \mathcal{F}_{s+1} \subseteq \dots$

associated graded complex $= g\widehat{CFK}(\mathcal{H}) = \bigoplus_s \mathcal{F}_s(\widehat{CFK}(\mathcal{H}))/\mathcal{F}_{s-1}(\widehat{CFK}(\mathcal{H}))$

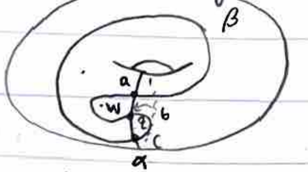
$\hat{\partial}_g : g\widehat{CFK}(\mathcal{H}) \to g\widehat{CFK}(\mathcal{H})$ kills all smaller Alexander grading ↗

$\hat{\partial}_g x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{(*)} \#\widehat{\mathcal{M}}(\varphi) y$ (only care abt part of $\hat{\partial}$ that preserves Alex gr)

$(*)$: $\varphi \in \pi_2(x,y)$, $\mu(\varphi) = 1$, $n_w(\varphi) = 0 = n_z(\varphi)$ $\boxed{(z/w \text{ like } X/O)}$

$\widehat{HFK}(K) = H_*(g\widehat{CFK}(\mathcal{H}))$

e.g.



trefoil

$\hat{\partial} a = 0$ no disc $b \to a$ b/c can't cross w

$\hat{\partial} b = c$ disk from $b \to c$, can cross z

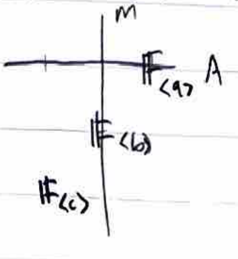
$\hat{\partial} c = 0$ ← $\widehat{CFK}(\mathcal{H})$ (max diff is even)

$A(b) - A(c) = 1$ $A(b) - A(a) = 1$ Abs. Alexander grading: make symmetrized

$g\widehat{CFK}(\mathcal{H})$: $\hat{\partial}_g a = 0$ $\hat{\partial}_g b = 0$ $\hat{\partial}_g c = 0$ | $\widehat{HF}(S^3) = \mathbb{F}_{(0)}$, a graded homology

$M(b) - M(c) = 1$ $M(b) - M(a) = -1$ | Absolute Maslov grading: $M(a) = 0$

$\widehat{HFK}(\mathcal{H}) = H_*(g\widehat{CFK}(\mathcal{H}))$



$$\begin{array}{ccccccccc} \dots = \mathcal{F}_{-2} & \subseteq & \mathcal{F}_{-1} & \subseteq & \mathcal{F}_0 & \subseteq & \mathcal{F}_1 & = & \mathcal{F}_2 = \dots \\ \parallel & & \parallel & & \parallel & & \parallel & & \\ 0 & & \langle c \rangle & & \langle b,c \rangle & & \langle a,b,c \rangle & & \\ & & & & \partial b = c & & \partial b = c & & \end{array}$$

$\widehat{HFK}$ detects genus (largest Alexander grading)

$\Delta_{T_{2,3}}(t) = t^{-1} - 1 + t$

- def: Osváth Szabó τ - invariant: $\tau := \min\{s : \iota : \mathcal{F}_s(\widehat{CFK}(\mathcal{H})) \to \widehat{CFK}(\mathcal{H})$ induces a surjection on $H_*\}$

eg: $\tau(T_{2,3}) = 1$ (H_* isn't surj until $s=1$)

- thm: $|\tau(K)| \leq g_4^{sm}(K)$
- Prop: $\widehat{CFK}(K_1 \# K_2) \cong \widehat{CFK}(K_1) \otimes_{\mathbb{F}} \widehat{CFK}(K_2)$ | exercise: given dbly ptd H.D. $\mathcal{H}_1, \mathcal{H}_2$ for K_1, K_2, find dp. H.D. for $K_1 \# K_2$

↳ Corollary: $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$

Hence: $\tau : \mathcal{C} \to \mathbb{Z}$ is a surjective homomorphism

- more flavors

$R = \mathbb{F}[U, V]$ bigraded ring $gr = (gr_U, gr_V)$ $gr(U) = (-2, 0)$, $gr(V) = (0, -2)$

↳ gr_U is Maslov grading as before

$CFK_R(\mathcal{H}) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_R$

$$\partial_R x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\varphi \in \pi_2(x,y),\ \mu(\varphi)=1} \#\hat{\mathcal{M}}(\varphi)\, U^{n_w(\varphi)} V^{n_z(\varphi)}\, y$$

relative gradings: $gr_U(x) - gr_U(y) = \mu(\varphi) - 2n_w(\varphi)$, $gr_V(x) - gr_V(y) = \mu(\varphi) - 2n_z(\varphi)$

$A(x) - A(y) = n_z(\varphi) - n_w(\varphi) = \frac{1}{2}(gr_U(x) - gr_U(y) - (gr_V(x) - gr_V(y))$

eg

$\partial a = 0$

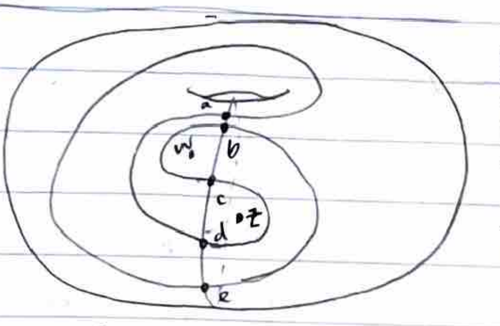
$\partial b = Vc + Ua$

$\partial c = 0$

depict:
$$\begin{tikzcd} Ua & b \arrow[l] \arrow[d] \\ & Vc \end{tikzcd}$$

$$\begin{array}{ccc} \cdots\ U^2 & U & 1 \\ & UV & V \\ & & V^2 \\ & & \vdots \end{array}$$

	gr_U	gr_V	A
a	0	-2	1
b	-1	-1	0
c	-2	0	-1



$\partial a = 0$

$\partial b = Va + Ve$

$\partial c = Ub + Vd$

$\partial d = Ua + Ue$

$\partial e = 0$

$$\begin{tikzcd} Ub \arrow[dd] \arrow[dr] & & c \arrow[ll] \arrow[dd] \\ & UVe & \\ UVa & & Vd \arrow[ll] \arrow[ul] \end{tikzcd}$$

change basis $\rightsquigarrow$

$$\begin{tikzcd} & Ub \arrow[d] & c \arrow[l] \arrow[d] \\ UVa & UV(a+e) & Vd \arrow[l] \end{tikzcd}$$

$(\Sigma, \alpha, \beta, z\, w)$ or

- $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ for $K \subset S^3$. then $(-\Sigma, \beta, \alpha, w, z)$ describes K^r
- $\mathcal{H}' = (-\Sigma, \beta, \alpha, z, w)$ describes K

$CFK_R(\mathcal{H})$ $CFK_R(\mathcal{H}')$

- same generators ———
- same differential but roles of w/z swapped (swap U/V, gr_U/gr_V) ⇝ chain homotopy equiv

$CFK_R(K)$ and concordance

- **thm** (Zemke): If $K_0 \sim K_1$ then $\exists$ absolutely gr_U, gr_V R-equivariant maps $CFK_R(K_0) \underset{g}{\overset{f}{\rightleftarrows}} CFK_R(K_1)$ s.t. f, g induce $\cong$ on $H_*(CFK_R(K_i)/U)/V\text{-torsion}$.
  ↳ $CFK_R(K)/U$ is a free f.g. $\mathbb{F}[V]$-module, so PID

  Fact: $H_*(CFK_R(K)/U) \cong \mathbb{F}[V] \oplus \bigoplus_{i=1}^{N} \mathbb{F}[V]/V^{n_i}$ exactly one free part for $K \subset S^3$

  rk: $H_*(CFK_R(K)/U) \cong H_*(CFK_R(K)/V)$ $[U \leftrightarrow V, gr_U \leftrightarrow gr_V]$
- **def**: $HFK^-(K) := H_*(CFK_R(K)/V)$ f.g. graded $\mathbb{F}[U]$-module
- **Prop**: $\tau(K) = -\frac{1}{2}\max\{gr_V(x) : x \in H_*(CFK_R(K)/U,\ V^n x \neq 0\ \forall n > 0\}$ (not V-torsion)
- alternating knots: $CFK_R(K)$ determined by $\Delta_K(t)$ and $\sigma(K)$
- **def**: $K \subset S^3$ is an L-space knot if $\exists r > 0$ st. $S^3_r(K)$ is an L-space
  ↳ $\mathbb{Q}HS^3$ Y is L-space if $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$ (generally $\geq$.)
- L-space knots: $CFK_R(K)$ determined by $\Delta_K(t)$.
- linear combinations of above: Künneth fmla: $CFK_R(K_1 \# K_2) = CFK_R(K_1) \otimes_R CFK_R(K_2)$

Alternating knots

- **thm** (Ozsváth-Szabó): Let K be alternating. Then $\widehat{HFK}(K)$ is supported in a single diagonal of slope 1 in Alex.-Maslov gr. plane $\overset{m}{+}_A$ and $\tau(K) = -\sigma(K)/2$
- **exercise**: for alternating, $CFK_R(K)$ is determined by $\Delta_K(t)$ and $\sigma(K)$ ↳ gives x-intercept

L-space knots

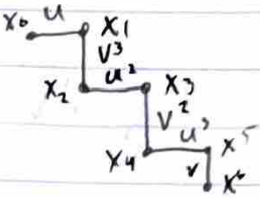
- **e.g.**: $\dim(\widehat{HF}(L(p,q))) = p$ (lens spaces are L-space)
  ↳ (p,q) torus knots are L-space knots:
  exercise: $S^3_{pq\pm1}(T_{p,q})$ is a lens space
- **thm** (O-S): If K is an L-space knot then $S^3_r(K)$ is an L-space $\forall r \geq 2g(K)-1$
- **thm** (O-S): If K is L-space then $\tau(K) = g(K)$, and $\widehat{HFK}$ is at most 1-dim in each A-grading
  Cor: If K is L-space then K fibered and nonzero coeffs of $\Delta_K(t)$ are ± 1.

  If K L-space: let $\Delta_K(t) = t^{n_0} - t^{n_1} + t^{n_2} - \ldots + t^{n_m}$ $(n_0 > n_1 > \ldots)$
  then $CFK_R(K)$ has $|\Delta_K(-1)|$ generators $x_0, \ldots, x_m$ and
  $\partial x_1 = U^{n_0 - n_1} x_0 + V^{n_1 - n_2} x_2$, $\partial x_3 = U^{n_2 - n_1} x_2 + V^{n_3 - n_4} x_4$, ...

- e.g.: $K = T_{4,5}$, $\Delta_K(t) = t^6 - t^5 + t^2 - 1 + t^{-2} - t^{-5} + t^{-6}$

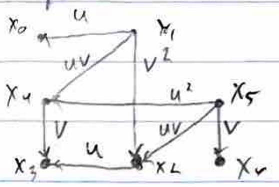
generators: $x_0, \dots, x_6$; $\partial x_0 = \partial x_2 = \partial x_4 = \partial x_6 = 0$,

$\partial x_1 = U x_0 + V^3 x_2$, $\partial x_3 = U^2 x_2 + V^2 x_4$, $\partial x_5 = U^3 x_4 + V x_6$



More concordance homs:

- **thm**: for each $k \in \mathbb{Z}_{>0}$, $\exists\ \varphi_k : \mathcal{C} \to \mathbb{Z}$ and $\bigoplus_k \varphi_k : \mathcal{C} \to \mathbb{Z}^\infty$ is surj (lin. indep)
  - ↳ e.g.: $K = T_{4,5}$, then ∂ info is contained in seq $(1, -3, 2, -2, 3, -1)$ (signed exponents in path from $x_0 \to x_6$, alternating U, V) (with $\to$ = $-$, against = $+$)
  - ↳ e.g.: $K = T_{2,3;2,1}$ ((2,1)-cable of $T_{2,3}$)



$\rightsquigarrow (1, -2, -1, 1, 2, -1)$

can we go backwards?
seq $\rightsquigarrow$ chain complex?
not exactly... but:

fix: set $UV = 0$ in ground ring. For a chain complex C over $\mathbb{F}[U,V]/UV$, let ∂_U be the induced boundary map on C/V, ∂_V ... on C/U.

- **def**: given a seq $(a_i)_{i=1}^{2N}$, $a_i \in \mathbb{Z} \setminus \{0\}$, the associated <u>standard complex</u> has generators $x_0, \dots, x_{2N}$ (over $\mathbb{F}[U,V]/UV$), differentials (↗ follow recipe above)

e.g.: $(-1, 3, 1, -1, -3, 1)$

$$x_0 \xrightarrow{U} x_1 \xleftarrow{V^3} x_2 \xleftarrow{U} x_3 \xrightarrow{V} x_4 \xrightarrow{U^3} x_5 \xleftarrow{V} x_6 \quad (\text{e.g. } \partial x_3 = U x_2 + V x_4)$$

- **thm**: every knot Floer complex over $\mathbb{F}[U,V]/UV$ has a standard complex as a direct summand, and it is unique (up to chain homotopy equiv).

Furthermore, the std. complex is a concordance invariant.

$[K \subset S^3 \rightsquigarrow CFK_{\mathbb{F}[U,V]}(K) \rightsquigarrow CFK_{\mathbb{F}[U,V]/UV}(K) \rightsquigarrow \text{standard complex} \underset{}{\overset{1-1}{\longleftrightarrow}} \text{sequence}]$

$\Rightarrow$ well-def set map $\mathcal{C} \to \{\text{sequences}\}$: $[K] \mapsto$ std complex seq.

grp. homo.?: "yes" [2 seqs → std complex, take tensor ⊗, use thm to put back into nice form]

e.g.: $C_1 = (1, -1)$ $\quad x_0 \xleftarrow{U} x_1 \xrightarrow{V} x_2$

$C_2 = (1, -1)$ $\quad y_0 \xleftarrow{U} y_1 \xrightarrow{V} y_2$ $\qquad C_1 \otimes_{\mathbb{F}[U,V]/UV} C_2$:

$$\begin{tikzcd}
x_0y_2 & x_1y_2 \arrow[l, "U"'] \arrow[r, "V"] & x_2y_2 \\
x_0y_1 \arrow[u, "V"] \arrow[d, "U"] & x_1y_1 \arrow[l, "U"'] \arrow[r, "V"] \arrow[u, "V"] \arrow[d, "U"] & x_2y_1 \arrow[u, "V"] \arrow[d, "U"] \\
x_0y_0 & x_1y_0 \arrow[l, "U"'] \arrow[r, "V"] & x_2y_0
\end{tikzcd}$$

change basis $\sim$

$$\begin{tikzcd}
x_0y_0 + x_2y_0 & x_1y_2 + x_2y_1 \arrow[l, "U"'] & x_2y_2 \\
x_0y_1 \arrow[u, "V"] & x_1y_1 \arrow[l, "U"'] \arrow[u, "V"] & x_2y_1 \arrow[u, "V"] \arrow[d, "U"] \\
x_0y_0 & x_1y_0 \arrow[l, "U"'] \arrow[r, "V"] & x_2y_0
\end{tikzcd}$$

$\rightsquigarrow (1, -1, 1, -1)$ ⟶ std complex summand

e.g.: $(1, -1) \otimes (2, -2) = (1, -1, 2, 1, -1, -2, 1, -1)$ ☹

inverse is make all terms negative

- <u>open</u>: give a description for the group operation on seqs
- <u>def</u>: given a finite seq (a_i) $(a_i \in \mathbb{Z} \setminus \{0\})$ and $j \in \mathbb{Z}_{>0}$, let $\varphi_j(a_i) = \#\{a_i = j \mid i \text{ odd}\} - \#\{a_i = -j \mid i \text{ odd}\}$ (signed count of how many $= j$)
- <u>thm</u>: φ_j is a homomorphism

$(\underline{1}, -1) \otimes (\underline{2}, -2) = (\underline{1}, -1, \underline{2}, 1, \underline{-1}, -2, \underline{1}, -1)$

$\varphi_j = \begin{cases} 1 & j=1 \\ 0 & \text{else} \end{cases}$ $\varphi_j = \begin{cases} 1 & j=2 \\ 0 & \text{else} \end{cases}$ $\varphi_j = \begin{cases} 1 & j=1,2 \\ 0 & \text{else} \end{cases}$ magic ☺

- <u>exercise</u>: $\varphi_j(T_{n,n+1}) = \begin{cases} 1 & j=1,\dots,n-1 \\ 0 & \text{else} \end{cases}$

↳ compute $\Delta_K(t)$, ~ CFK, use defs

<u>Cor</u>: $\bigoplus_j \varphi_j : \mathcal{C} \to \mathbb{Z}^\infty$ is surjective

- Consider: $0 \to \mathcal{C}_{TS} = \{\text{top. slice knots}\} \to \mathcal{C}_{sm} \to \mathcal{C}_{top} \to 0$ ker

<u>e.g.</u> $Wh(K) \in \mathcal{C}_{TS}$ $\forall K$ by Freedman since $\Delta_{Wh(K)}(t) = 1$

then take any not sm slice knot: $Wh(T_{2,3})$

- <u>thm</u>: $\mathcal{C}_{TS}$ contains a $\mathbb{Z}^\infty$ direct summand

↳ Ozsváth-Stipsicz-Szabó originally proved using conc. homs Υ_K upsilon

reproof with φ_j: <u>thm</u>: $\bigoplus_j \varphi_j : \mathcal{C}_{TS} \to \mathbb{Z}^\infty$ surjective

<u>Pf</u>: $D = Wh(T_{2,3})$. D top slice $(D \sim_{top} U) \Rightarrow D_{n,n+1} \sim_{top} U_{n,n+1} = T_{n,n+1}$ $(n,n+1)$-cable

$\Rightarrow D_{n,n+1} \# -T_{n,n+1} \sim_{top} U$.

<u>claim</u>: $\varphi_j(D_{n,n+1}) = \{ n \ \ j=1;\ 1 \ \ 1<j<n-1, j=n;\ 0 \ \ j=n-1, j>n$

Claim + exercise $(\varphi_j(T_{n,n+1})) \Rightarrow \varphi_j(D_{n,n+1} \# -T_{n,n+1}) = \{1 \ j=n, \ 0 \ j>n$ □

- If K L-space, then (a_i) (from std complex) determined by $\Delta_K(t)$
- <u>thm</u> (Hedden, Hom): let $p>0$. $K_{p,q}$ is an L-space knot iff K is L-space and $q > p(2g(K)-1)$ (classifying cabling)

<u>Pf</u> $(\Leftarrow)$: <u>Claim</u>: $S^3_{pq}(K_{p,q}) = S^3_{q/p}(K) \# L(p,q)$.

<u>Pf</u>: for a knot $J \subset S^3$, let $E(J) = S^3 - \nu(J)$, $T_J = \partial(\nu(J))$.

$E(K_{p,q}) = \underset{\nearrow}{E(K)} \underset{\underset{\uparrow}{T_K - A}}{\cup} \underset{\nwarrow}{\nu(K)}$, $A = \nu(K_{p,q}) \cap T_K = \bigcirc$

knot complement white = annulus solid torus inside

(knotted as K)

now glue solid torus $S^1 \times D^2$ to $E(K_{p,q})$ s.t. $\{0\} \times \partial D^2$ maps to pq-framed longitude λ of $K_{p,q}$.

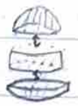
exercise: λ is the surface framing of $K_{p,q}$ on T_K.

(draw $K_{p,q}$ on unknotted torus, pushoff, compute lk #)

decompose $S^1 \times D^2 = ([0,\pi] \times D^2) \cup ([\pi, 2\pi] \times D^2)$ 

exercise: 1) $E(K) \cup ([0,\pi] \times D^2) = S^3_{q/p}(K) - B^3$

2) $\nu(K) \cup ([\pi, 2\pi] \times D^2) = L(p,q) - B^3$

3) $(\{0\} \times D^2) \cup (T_K - A) \cup (\{\pi\} \times D^2)$  $\square$ (claim)

So if K is L-space and $q/p \geq 2g(K) - 1$, then $S^3_{q/p}(K)$ is L-space. $L(p,q)$ is L-space and # of L-space is L-space (b/c $\widehat{HF}(Y_1 \# Y_2) = \widehat{HF}(Y_1) \otimes \widehat{HF}(Y_2)$).

Thus $S^3_{pq}(K_{p,q}) = S^3_{p/q}(K) \# L(p,q)$ is L-space $\square$

↳ e.g.: $p > 0$. $T_{2,3;p,q}$ is L-space knot iff $q > p(2g(T_{2,3}) - 1) = p$

- exercise: find (a_i) for $T_{2,3;n,n+1}$

↳ Recall: $\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t)$, $\Delta_{T_{p,q}}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$

[Goal: $\varphi_j(D_{n,n+1})$, $D = Wh(T_{2,3})$]

- Prop: If K_0, K_1 have the same std. complex rep, then $P(K_0), P(K_1)$ have same std complex rep

↳ satelliting give a well-def map on seq's (a_i).

- Prop: $D = Wh(T_{2,3})$ has same std. complex as $T_{2,3}$ (i.e. $(1,-1)$).

$\Rightarrow$ So $\varphi_j(D_{n,n+1}) = \varphi_j(T_{2,3;n,n+1})$

more conc. invariants?

- def: let (a_i) be seq for K. $\varepsilon(K) = \{1 \ a_1 > 0, \ -1 \ a_1 < 0, \ 0 \ a_1 = 0$ (trivial seq)

$\varepsilon : \mathcal{C} \to \{-1, 0, 1\}$ not homo (range is not a gp) (but still ε and # ?)

- Prop:

$\varepsilon(K_0)$	$\varepsilon(K_1)$	$\varepsilon(K_0 \# K_1)$
+1	+1	+1
-1	-1	-1
0	±1	±1
+1	-1	anything

Observe: $K \rightsquigarrow CFK(K)$

$-K \rightsquigarrow CFK(K)^*$ (dual)

$K_1 \# K_2 \rightsquigarrow CFK(K_1) \otimes CFK(K_2)$

K slice $\Rightarrow \varepsilon(CFK(K)) = 0$

ε-equivalence / local equiv over $\mathbb{F}[U,V]$

- def: $\mathfrak{CFK} := \left(\left\{ \begin{array}{l} \text{complexes over } \mathbb{F}[U,V] \\ \text{that satisfy same alg. prop as} \\ CFK(K) \text{ (like knot Floer complex)} \end{array} \right\} / \sim, \otimes \right)$ $C_0 \sim C_1 \Leftrightarrow \varepsilon(C_0 \otimes C_1^*) = 0$ is a group

and $\mathcal{C} \to \mathfrak{CFK} : [K] \mapsto [CFK(K)]$ is a gp. homo (open: surjective?)

and $CFK(K_0) \sim CFK(K_1)$ iff K_0, K_1 have same seq.

Tye Lidman

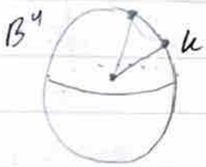
<u>Q</u>: What surfaces can a knot bound in B^4?

↳ sm slice knots bound sm. embedded disks, top slice knots bound top. locally flat ~~embedded~~ disks

↳ sm embedded, compact, orientable surfaces (Seifert surface)

↳ all knots bound immersed disks (nullhomotopy of knot, $\mathbb{R}^3$ contractible)

<u>Q</u>: What about embedded disks?



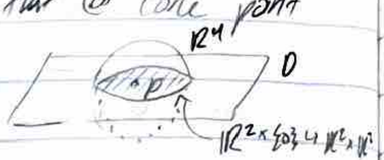
$B^4 = \text{Cone}(S^3)$

$\text{Cone}(K)$ is an embedded disk in B^4

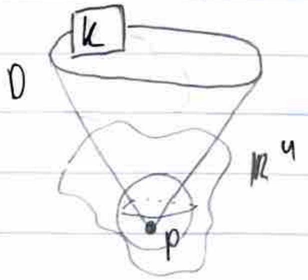
(not nec. top. locally flat).

<u>Prop</u>: If K nontrivial, then the disk is not locally flat @ cone point

<u>Pf</u> (Sketch): If D is a disk that is locally flat @ p, then there is an atlas that is locally product



$$\pi_1(\mathbb{R}^4 \setminus D) = \pi_1(\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})) = \mathbb{Z}.$$



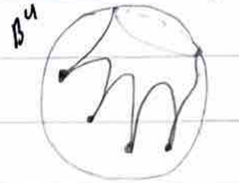
$\mathbb{R}^4 \setminus D$ retracts onto $B^4 \setminus \text{Cone}(K) = (S^3 \setminus K) \times I$

$$\pi_1(\mathbb{R}^4 \setminus D) \twoheadrightarrow \underbrace{\pi_1(S^3 \setminus K)}_{\text{nonabelian \& never } \mathbb{Z} \text{ if } K \text{ is nontrivial}}$$

so $\pi_1(\mathbb{R}^4 \setminus D) \neq \mathbb{Z}$.

Hence $\text{Cone}(K)$ is not locally flat if K nontrivial. □

<u>def</u>: a disk in B^4 is a <u>PL-disk</u> if it is sm. embedded except at some singularities that are $\text{Cone}(K) \subseteq B^4$.

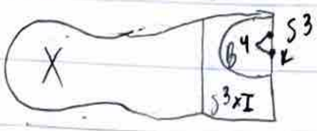


↳ all knots in S^3 bound PL-disks

can replace PL-disk w/ one with only one singularity

<u>Q</u>: If $K \subset S^3$ and X^4 sm. w/ $\partial X = S^3$ does K bound a PL-disk in X?

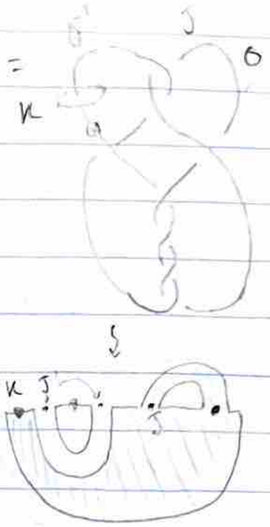
Yes: use collar of ∂X to find B^4 containing K



<u>Conjecture</u> (Zeeman's Conjecture): $\exists$ contractible 4-mfd Z w/ knot $K \subset \partial Z$ st. K cannot bound a PL-disk in Z.

<u>Thm</u> (Akbulut): Conjecture is true

Pf: $Z =$



2 comp. link:

$J'\ \bigcirc$ = remove disc the unknot bounds from B^4

$J\ \bigcirc^{0}$ = attach 2-handle $D^2 \times D^2$ to B^4 along nbhd$(J) = S^1 \times D^2$ so that $S^1 \times \{1\}$ goes to λ_J

exercise: Z contractible (compute π_1 and homology).

(omitted): K doesn't bound PL-disk in Z. $\square$

(∂Z is 3-mfd obtained from 0-surgery on both components.)

exercise: find a different 4-mfd Z' with $\partial Z = \partial Z'$ where K bounds sm embedded disk in Z'.

↳ swap • and ○ decorations on links (the Z and Z' are diffeo)

Conjecture (Modified Zeeman): $\exists\ Y^3 = \partial$ contractible 4-mfd and $K \subseteq Y$ which can't bound PL-disk in any contractible 4-mfd with boundary Y

thm (Levine): Conjecture is true

Pf: $Y = S^3_{-1/2}$ (R.H.T.), K = core curve of solid torus used in surgery

Fact: Y bounds contractible 4-mfd.

[WTS: K can't bound PL-disk in any $\mathbb{Z}HB^4$]

Sps K bounds PL disk D in $\mathbb{Z}HB^4$ Z with one cone singularity (Cone$(J \subseteq S^3)$)

$Z \setminus$ nbhd(cone pt): cobordism $S^3 \to Y$; $D \setminus$ nbhd(cone pt): $J \to K$ sm. knot concordance inside $Z \setminus$ nbhd(cone pt)

$\Rightarrow S^3_{1/n}(J) \underset{\text{hom. cob.}}{\sim} Y_{1/n}(K)$ $\forall n$

[**Recall**: d invariant. 1) $d(Y_1) = d(Y_2)$ if Y_i homology cob. 2) $d(S^3_{1/n}(Q))$ has fmla.]

$d(S^3_{1/2}(J)) \overset{(2)}{=} d(S^3_{1/3}(J))$

$d(Y_{1/n}(K)) = d(S^3_{1/n-2}(RHT))$

$d(Y_{1/2}(K)) = 0$. $d(Y_{1/3}(K)) = d$(Poincaré hom. sphere) $= -2 \neq 0$ ↯ $\square$

Tye Lidman 2

I. Dehn Surgery

$S^3_{1/0}(K) = S^3$. $S^3_{p/q}(U) = L(p,q)$, $S^3_{1/n}(U) = S^3 \; \forall n$, $S^3_{2/n}(U) = \mathbb{RP}^3 \; \forall n$ odd

Q: how injective is surgery wrt p/q?

thm (Gordon-Luecke): $S^3_{p/q}(K) = S^3_{1/0}(K) \Rightarrow (K = U$ and $p/q = \frac{1}{n})$ or $(p/q = 1/0)$

Cosmetic Surgery Conjecture: If $S^3_{p/q}(K) \cong_+ S^3_{p'/q'}(K)$, then $p/q = p'/q'$ or $K = U$

What's known? Assume $S^3_{p/q}(K) = S^3_{p'/q'}(K)$, $p/q \neq p'/q'$, $K \neq U$

1) $H_1(S^3_{p/q}(K)) = \mathbb{Z}_p \Rightarrow |p| = |p'|$

(Futer-Purcell-Schleimer)

2) hyperbolic geometry $\Rightarrow$ can enumerate possible $\{p/q, p'/q'\}$ for given K

3) Heegaard-Floer: "$\dim HF(S^3_{p/q}(K)) = |p| + |q| \, c(K)$, $c(K) = 0 \Leftrightarrow K = U$

(Ozsváth-Szabó, Wang, Ni-Wu): $p'/q' = -p/q$

(Hanselman): $\{p/q, p'/q'\} = \{1/n, -1/n\}$ or $\{-2, +2\}$

Unfortunately, $HF(S^3_{1/1}(9_{44})) \cong HF(S^3_{-1/1}(9_{44}))$

thm: [1]: If counterexample, then

a) $\{p/q, p/q'\} = \{2, -2\}$ b) $g(K) = 2$ c) $\Delta_K(t) = 1$

$\Rightarrow$ C.S.C. holds for alternating, fibered, non top. slice. knot

thm (Ren): CSC holds $\Leftrightarrow$ holds for hyperbolic knots

thm: [2]: let $K \subseteq S^2 \times S^1$, $K \neq \{pt\} \times S^1$ and generates π_1 (winding # 1)

then $Y_n \not\cong_+ Y_m$ if $n \neq m$  $\overset{n}{:}$ surgery $= Y_n$

II. Instantons

① $Y = \mathbb{Z}HS^3 \rightsquigarrow I_*(Y)$ $\mathbb{Z}$ graded + $\mathbb{R}$-filtration → new from HF.

② $W: Y_1 \to Y_2$ cobordism $\rightsquigarrow I_*(W): I_*(Y_1) \to I_*(Y_2)$

- $I_*(W)$ is filtered

$I_*(S^3) = 0$

- If $\pi_1(W) = 0 \Rightarrow I_*(W)$ lowers filtration ↗

meta **thm**: If $W: Y_1 \to Y_2$ w/ 1) $\pi_1(W) = 0$ and $b_2^+ = 0$ 2) $I_*(Y_1) \neq 0$

3) $I_*(W)$ is iso then $Y_1 \not\cong_+ Y_2$

Pf: $I_*(Y_1) \cong I_*(Y_2)\{\text{filtration shift}\} \not\cong I_*(Y_2)$ ◻

③ $\exists$ exact triangle
$$\begin{tikzcd} I_*(Y) \arrow[rr, "(\cdot)"'] & & I_*(Y_{-1}(K)) \arrow[dl] \\ & I_*'(Y_0(K)) \arrow[ul] & \end{tikzcd}$$

$(\cdot)$: $I(W_{-1}(K))$ 2-handle cobordism

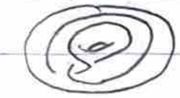
Pf of [2]: use exact triangle & meta thm. $Y_{n_{-1}}(J) \cong Y_{n+1}$

$Y_{n_0}(J) \cong S^2 \times S^1$

end ————

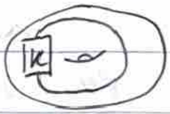
- **Recall**: $\Delta_{P(K)}(t) = \Delta_K(t^w) \cdot \Delta_{P(U)}(t)$ (w = winding #)
  $g(P(K)) = |w|\, g(K) + g(P \subset S^1 \times D^2)$
- **Thm**: $\tau(K_{p,q})$ ((p,q)-cable) depends on $p, q, \tau(K), \varepsilon(K)$
  1) If $\varepsilon(K)=1$: $\tau(K_{p,q}) = p \cdot \tau(K) + (p-1)(q-1)/2$
  2) If $\varepsilon(K)=-1$: $\tau(K_{p,q}) = p \cdot \tau(K) + (p-1)(q+1)/2$
  3) If $\varepsilon(K)=0$: $\tau(K)=0$ and $\tau(K_{p,q}) = \tau(T_{p,q}) = \left\{ \frac{(p-1)(q-1)}{2} \ q>0, \ \frac{(p-1)(q+1)}{2} \ q<0 \right.$
- **Recall**: $P : \mathcal{C} \to \mathcal{C} : [K] \mapsto [P(K)]$ is well def (as a set map)

Q: When is this injective, surjective, bijective?

e.g: $P =$  $=$ Wh. $\Delta_{Wh(K)}(t) = 1 \Rightarrow Wh : \mathcal{C} \to \mathcal{C}$ is not surjective
$\mathcal{C}$ Fox-Milnor condition

OR: $g(Wh(K)) = 1$ but being concordant to genus 1 means slice genus ≤ 1.

exercise: If $w(P) \neq \pm 1$ then P is not surjective

e.g: $P =$  is surjective (same as $\# K$, inverse is $\# -K$)

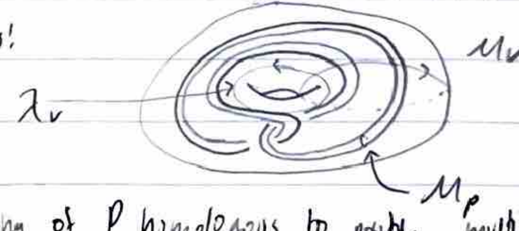
e.g: any pattern concordant to in $(S^1 \times D^2) \times I$

$Q =$ 

Thm (Levine): The Mazur pattern Q is not surjective on $\mathcal{C} \to \mathcal{C}$
$\hookrightarrow$ Prop: $\tau(Q(K)) = \{ \tau(K)$ if $\tau(K) \leq 0$ & $\varepsilon(K) = 0, 1$ $\quad \tau(K)+1$ if $\tau(K)>0$ or $\varepsilon(K)=-1$
and $\varepsilon(Q(K)) = \{ 0$ if $\tau(K) = \varepsilon(K) = 0 \quad 1$ else (≥ 0 so not surj)

Q: bijective P? (that are not $\#$) A: Yes!

[Miller - Piccirillo "Knot traces and concordance"]



Consider $P \subset S^1 \times D^2 =: V$, thicken it

take λ_P (longitude for thickened) = Unique framing of P homologous to positive multiple of λ_V in $V - \nu(P)$ = Seifert framing of $P(U)$

def: A pattern $P \subset V$ is <u>dualizable</u> if $\exists\, P^* \subset S^1 \times D^2 =: V^*$ s.t. $\exists$ orientation reversing homeo $h : V - \nu(P) \to V^* - \nu(P^*)$ with $h(\lambda_V) = \lambda_{P^*}$, $h(\mu_V) = -\mu_{P^*}$, $h(\lambda_P) = \lambda_{V^*}$, $h(\mu_P) = -\mu_{V^*}$. P^* is the <u>dual of P</u>.

Given any embedding $D^2 \to S^2$, we get an embedding $S^1 \times D^2 \to S^1 \times S^2$.
Hence $P \subset S^1 \times D^2$ induces a knot $\hat{P} \subset S^1 \times D^2$. (equivalently, view $S^1 \times D^2$ as one of the genus 1 handlebodies in a Heegaard splitting of $S^1 \times S^2$)

$S^1 \times \{pt\}$

<u>Prop</u>: $P \subset V$ is dualizable iff $\hat{P}$ is isotopic to $\hat{\lambda}_V$ in $S^1 \times S^2$ (the S^1 factor)

<u>Pf</u>: $(\Leftarrow)$. let $V^* = (S^1 \times S^2) - \nu(\hat{\lambda}_V) = (S^1 \times S^2) - \nu(\hat{P})$

<u>exercise</u>: $P^* = \hat{\lambda}_V \subset V^*$

$(\Rightarrow)$: $M := S^1 \times S^2 - \nu(\hat{P})$ = Dehn filling of $V - \nu(P)$ along μ_V

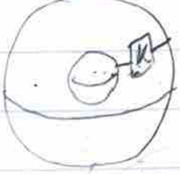
P dualizable $\Rightarrow$ $M \cong$ Dehn filling of $V^* - \nu(P^*)$ along μ_{P^*} $= V^*$.

thus $\hat{P}$ is a knot in $S^1 \times S^2$ with solid torus complement $\Rightarrow$ $\hat{P}$ isotopic to $\pm\lambda_V$ □

↳ (like how U is unique knot in S^3 w/ solid torus complement)

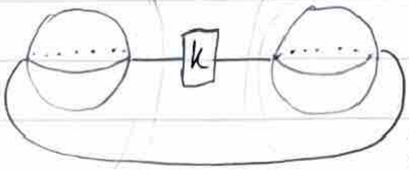
<u>eg</u> $K_\# =$

$\hat{K}_\# =$ (in $S^1 \times S^2$)



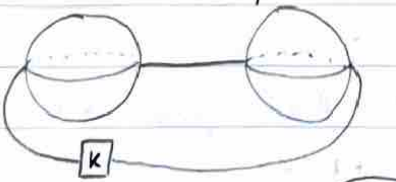
$I \times S^2$ (thickened ball/shell) identify inside/outside

or

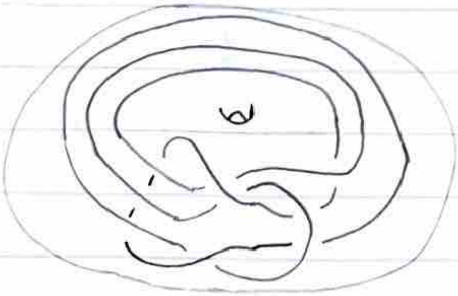


identify left/right

= pull sphere along K

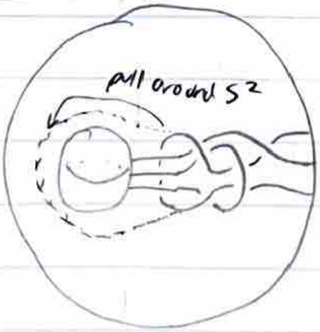


<u>eg</u>:

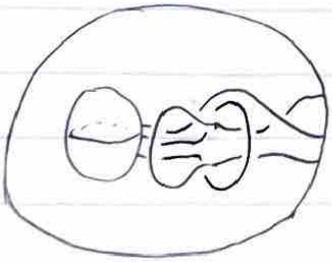


in $S^1 \times S^2$

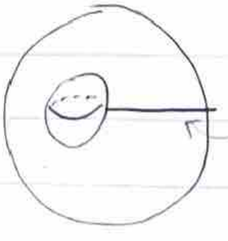
$\rightsquigarrow$



$\rightsquigarrow$



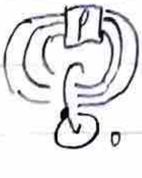
$\rightsquigarrow$ isotopy



S^1 factor so dualizable

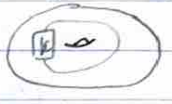
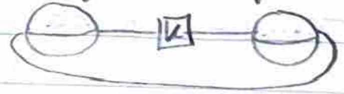
- <u>thm</u>: $X_0(P(U)) \cong X_0(P^*(U))$

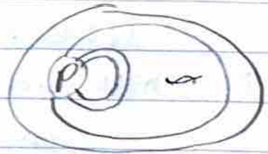
<u>Pf idea</u>:



<u>exercise</u>: 1-handle and either 2-handle give B^4. Remaining 2-handle is attached along either $P(U)$ or $P^*(U)$

- **Prop**: If P, Q dualizable, then $P \circ Q$ is dualizable w/ dual $Q^* \circ P^*$. Pf: exercise
- Let $\overline{P}$ be the pattern obtained from $P \subset S^1 \times D^2 = V$ by reversing orientation of V and string orientation (change crossings & arrow)

$K_\# =$  dual is also $K_\#$  : 

$P_\# := P(U)_\#$ 

- **Cor**: $\overline{P^*}(P(U)) \sim U \sim P(\overline{P^*}(U))$

Pf: $X_0(\overline{P^*}(P(U))) = X_0(\overline{P^*} \circ P_\#(U)) \overset{(\text{thm})}{=} X_0((\overline{P^*} \circ P_\#)^*(U)) \overset{(\text{Prop})}{=} X_0(\overset{\overset{P_\#}{\shortparallel}}{P_\#^*} \circ \overline{P}(U))$
$= X_0(P(U) \# \overline{P}(U))$ is slice. By Trace Embedding Lemma, $X_0(P(U) \# \overline{P}(U))$ embeds in $S^4 \Rightarrow X_0(\overline{P^*}(P(U)))$ embeds in $S^4 \Rightarrow$ By TEL, $\overline{P^*}(P(U))$ is slice. (other same) □

- **thm** (Miller-Piccirillo, reproof of Gompf-Miyazaki): $\overline{P^*}(P(K)) \sim K \sim P(\overline{P^*}(K))$

Pf. $\overline{P^*}(P(K)) \# -K = (\overline{K_\#} \circ \overline{P^*})(P \circ K_\#)(U) = (\overline{K_\# \circ P^*})(P \circ K_\#)(U)$
$K_\# \circ P^*$ is dual of $P \circ K_\#$ so by prev Cor, it's slice. □

- **thm**: $\exists$ inf. many pairs of knots K_n, K_n' w/ diffeomorphic 0-trace, but K_n, K_n' are not concordant, even up to orientation reversal

exercise: for any dualizable P, adding full twists $\tau_n(P)$ is dualizable w/ dual $\tau_{-n}(P^*)$

Pf: $P =$  $\tau_n(P) =$ add full twists. $K_n = \tau_{2n-1}(P)$
$K_n' = \tau_{-2n-3}(P)$

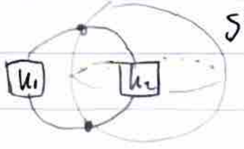
exercise: $X_0(K_n) \cong X_0(K_n')$

$K_n \not\approx K_n'$: double branched covers of S^3 branched over K

Prop: If K slice, then $\Sigma_2(K)$ bounds a $\mathbb{Q}HB^4$

Pf: Exercise: $D \subset B^4$ slice disk of K, then d.b.c. of B^4 branched over D is $\mathbb{Q}HB^4$ w/ boundary $\Sigma_2(K)$ □

Prop: $\Sigma_2(K_1 \# K_2) = \Sigma_2(K_1) \# \Sigma_2(K_2)$

Pf: 
S^2 d.b.c of S^2 branched over 2 pts is S^2 □

↑ $\mathbb{Q}$ homology cob. gp

So we have homo: $\mathcal{C} \to \Theta^3_{\mathbb{Q}} : [K] \mapsto [\Sigma_2(K)]$

exercise: $|H_1(\Sigma_2(K);\mathbb{Z})| = |\Delta_K(-1)|$

↳ -1 is primitive 2nd root of unity, double branched cover

★ invariants of $\mathbb{Q}$-hom. cob. to obstruct concordance of knots

Fact: If W is $\mathbb{Q}H_*$ cob. between $\mathbb{Z}HS^3$ Y_0, Y_1, then $d(Y_0) = d(Y_1)$

Prop: $\Sigma_2(K_n)$, $\Sigma_2(K_n')$ are $\mathbb{Z}HS^3$ but $d(\Sigma_2(K_n)) = 0$, $d(\Sigma_2(K_n')) = -2$

exercise: Use Prop? to show $p: \mathcal{C} \to \mathcal{C}$ is not connect sum

Open

1) Slice-Ribbon conjecture: Is every slice knot ribbon?
2) Smooth 4D Poincaré conjecture: Is every sm, closed 4-mfd homotopy equiv to S^4, diffeo to S^4?
3) does $\exists$ not slice knot $K \subset S^3$ st. K bounds a sm slice disk in
a) homotopy B^4? b) $\mathbb{Z}HB^4$?
note: (a) gives disproof of sm 4-Poincaré conjecture
note: $\exists$ not slice knots that do bound sm slice disks in $\mathbb{Q}HB^4$ (e.g. 4_1)
↳ more generally, any strongly negatively amphichiral knot: $\exists$ orientation reversing homeo $\varphi: S^3 \circlearrowleft$ st. $\varphi(K) = K$ and φ has exactly 2 fixed pts on K
4) Consider $\ker(\Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}})$, i.e. $\mathbb{Z}HS^3$ which bound $\mathbb{Q}HB^4$ (e.g. $\Sigma(2,3,7)$). Is ker inf. generated?
5) Ribbon concordance from K_0 to K_1 is a concordance w/ no local max.
Conjecture (Gordon): Ribbon concordance is a partial order. Agol: Yes.
- Q: Fixing K, what can we say about the poset $[K]$?
exercise: If $K_0 \sim K_1$, then $\exists K_2$ st. $K_0 < K_2$, $K_1 < K_2$.
↳ take concordance, rearrange min, max, saddles.
- Q: Is the order type of $[K]$ indep of K?
- Q: $\exists$ infinite descending chain $K_0 > K_1 > K_2 > \ldots$?
- Q: does every concordance class contain a unique minimal elt?

6) $\exists$ torsion in $\Theta^3_{\mathbb{Z}}$? (hard part is showing 2-torsion is nontrivial)
7) Is all torsion in $\mathcal{C}$ generated by negatively amphichiral knots?