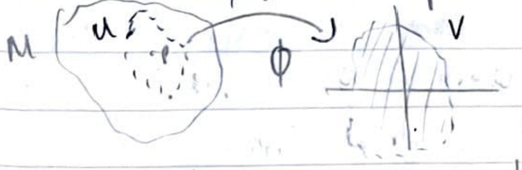
Smooth Manifolds

- def: a topological space M is a manifold of dim n (n-manifold) if
1) M is Hausdorff and 2^{nd} countable
2) M is locally Euclidean (each pt $p \in M$ has an open nbhd $p \in U \subseteq M$, open $V \subseteq \mathbb{R}^n$, $\exists$ homeomorphism $\phi : U \to V$)



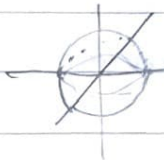
↳ called coordinate chart

Hausdorff: distinct pts can be seperated by open sets

$U \cap V = \emptyset$ (seq conv to unique p)

second ctbl: topology on M has ctbl. basis

- can replace 1) w/ $M \subseteq \mathbb{R}^k$
- ex: show in 2) that we can take $V \subseteq \mathbb{R}^n$ to be an open ball and $\phi(p) = 0$ OR $V = \mathbb{R}^n$
- ex: S^2 is a manifold



U_{z+} = upper hemisphere $V_{z+} = \{(x,y) \in \mathbb{R}^2, x^2+y^2 < 1\}$
$U_{z+} \to V_{z+}$ by $(x,y,z) \to (x,y)$
projection map is continuous, so is inverse $\Rightarrow$ coordinate chart
do same for lower, east/west, back/front hemisphere

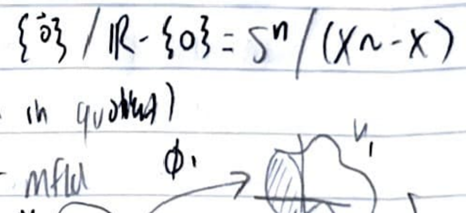
$S^2 \subseteq \mathbb{R}^3$ and $\mathbb{R}^3$ is Hausdorff & 2^{nd} ctbl

Way 2: stereographic projection

- ex: $\mathbb{RP}^n = \{$ lines through origin in $\mathbb{R}^{n+1}\} = \mathbb{R}^{n+1} - \{\vec{0}\} / \mathbb{R} - \{0\} = S^n / (x \sim -x)$

★? homogenous coordinates denoted $[x^0 : \ldots : x^n]$ (pts. identified in quotient)

- Product of n-manifold and m-manifold is a $(m+n)$-mfld
- ex: $\mathbb{CP}^n$ is a $2n$-mfld
- def: given 2 coord charts $\phi_1 : U_1 \to V_1$, $\phi_2 : U_2 \to V_2$, they are smoothly compatible if

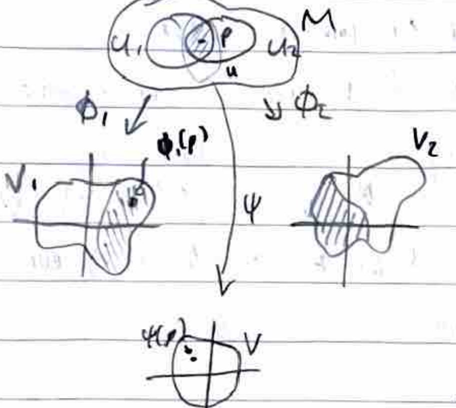


$\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ are both smooth maps (i.e., all partial derivs of all orders exist)

- def: a smooth atlas is a collection of coord charts $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in I}$ st. 1) $\{U_\alpha\}$ cover M 2) all coord charts are smoothly compatible
- lemma 1: M n-manifold. 1) every smooth atlas is contained in some maximal smooth atlas 2) two smooth atlases for M determine the same maximal sm. atlas iff their union is a smooth atlas

x^3
↓

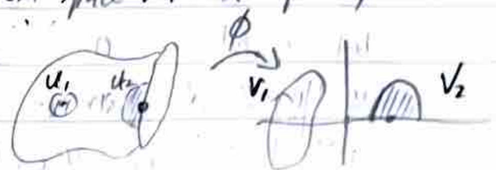
Pf: 17) given $A = \{\Phi_\alpha\}$ smooth atlas, let $\bar{A} = A \cup \{$all coord charts sm. compatible w/ elts of $A\}$, claim $\bar{A}$ is a smooth atlas. It obv. still covers M. For the other condition, let $\phi_1, \phi_2 \in \bar{A}$.
WTS: $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ (and $\phi_1 \circ \phi_2^{-1}$) smooth



(show it's smooth at every pt in $\phi_1(U_1 \cap U_2)$)
take $\phi_1(p)$ in $\phi_1(U_1 \cap U_2)$. $\exists$ some $\psi : U \to V$ in A (it's a smooth atlas, covers M) with $p \in U$. Then
$\phi_2 \circ \phi_1^{-1} = (\phi_2 \circ \psi)^{-1} \circ (\psi \circ \phi_1^{-1})$
smooth @ $\psi(p)$ by def of $\bar{A}$; sm @ $\phi_1(p)$ by def of $\bar{A}$
so ϕ_1, ϕ_2 are sm. compatible ✓

- def: a smooth manifold M is a manifold w/ a maximal smooth atlas (but by lemma, any smooth atlas gives a max sm. atlas)
  - max smooth atlas $\approx$ smooth structure
- def: $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is called a transition function or coordinate transformation
- examples: S^n has smooth struc, $\mathbb{R}P^n$, any open set U in a sm manifold M (with sm atlas A) is a sm mfld (with atlas restricted coord charts to U)
- $M(n, m; \mathbb{R}) = n \times m$ matrices, entries in $\mathbb{R} = \mathbb{R}^{m \cdot n}$ (has sm structure $\{id : \mathbb{R}^{mn} \to \mathbb{R}^{mn}\}$)
- $GL(n, \mathbb{R}) = \{$invertible $n \times n$ matrix in $\mathbb{R}\} = \det^{-1}(\mathbb{R} - \{0\})$
  Claim: det is cont, so $\det^{-1}(\mathbb{R} - \{0\})$ is open and $GL(n, \mathbb{R})$ has sm structure of dim n^2
  Pf: $(n=2)$: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ is poly in $(a,b,c,d) \Rightarrow$ smooth $\Rightarrow$ cont ✓
  (Induction step): cofactor expansion to get det $n \times n \to$ det of $(n-1)\times(n-1)$ is cont ✓ □
- sus example: $M = \mathbb{R}$, $A = \{id : \mathbb{R} \to \mathbb{R}\}$, $B = \{f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3\}$
  B not compatible w/ A ($x \mapsto x^{1/3}$ is a homeo but not smooth @ $x = 0$)
- def: two smooth mfolds (M, A), (N, B) are diffeomorphic if $\exists$ a homeomorphism $f : M \to N$ s.t. $\phi \in B \Leftrightarrow \phi \circ f \in A$
  ↳ converts elts in one max'l atlas to another max'l atlas
  ↳ essentially the same

- def: an n-manifold w/ boundary is a Hausdorff, 2nd ctbl space M s.t. $\forall p \in M$, $\exists$ 1) open set $U \subseteq M$, $p \in U$ 2) open $V \subseteq \mathbb{R}^n_{\geq 0}$
3) homeomorphism $\phi : U \to V$



$\{(x^1, \dots, x^n) \mid x^n \geq 0\}$
only last coordinate is ≥ 0

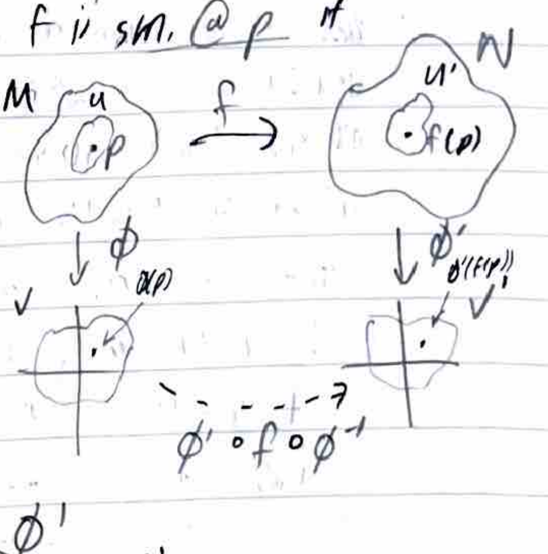
- def: $\mathrm{int}(M)$ = interior = $\{p \in M$ with coord. chart $\phi : U \to V$ s.t. $\phi(p)$ has positive x^n coord$\}$
∂M = boundary = $\{p \in M$ with coord chart $\phi: U \to V$ & $\phi(p)$ has 0 x^n coord$\}$

▷ 1) ∂M is a $(n-1)$ manifold (all pts get mapped to 0 x^n coord)
2) $\mathrm{int}\, M$ is a n-manifold w/o boundary
3) $\partial(\partial M) = \emptyset$
4) $\partial(\mathrm{int}\, M) = \emptyset$
5) $\mathrm{int}(\partial M) = \partial M$
6) $(\mathrm{int}\, M) \cap (\partial M) = \emptyset$

- def: If $A \subseteq \mathbb{R}^n$ is any set and $f : A \to \mathbb{R}^k$, then f is smooth if $\forall p \in A$, $\exists U \supseteq A$ open and smooth $F : U \to \mathbb{R}^k$ s.t. $f|_{A \cap U} = F|_{A \cap U}$ — extend f to open set
↳ now can define sm. str. on manifolds w/ boundary as before

- def: let M, N be manifolds, $f: M \to N$ map. f is sm. @ p if $\exists$ 1) coord chart $\phi : U \to V$ about $p \in M$ 2) a coord chart $\phi' : U' \to V'$ about $f(p) \in N$ s.t. $\phi' \circ f \circ \phi^{-1}$ is sm. @ $\phi(p)$

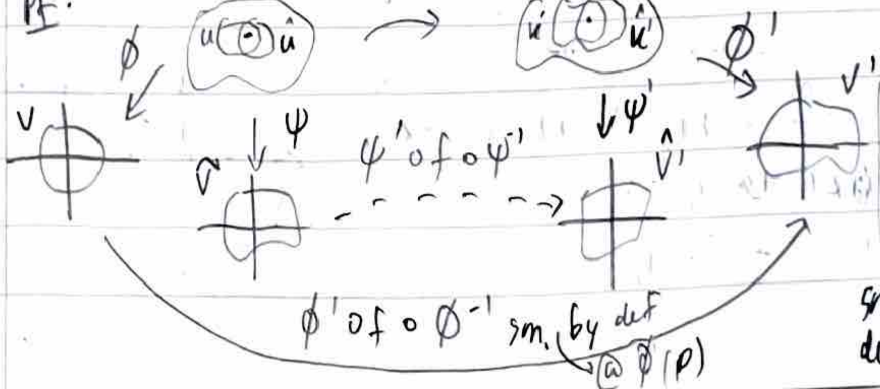
only need to check 1 chart pair



- lemma 2: If f is sm @ p, then for any coord charts $\psi : \hat{U} \to \hat{V}$ abt p, $\psi' : \hat{U}' \to \hat{V}'$ abt $f(p)$, we have $\psi' \circ f \circ \psi^{-1}$ is smooth @ $\psi(p)$

it is true for any charts

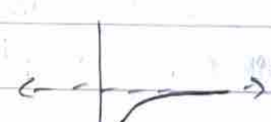
Pf:



$\psi' \circ f \circ \psi^{-1}$
$= \psi' \circ (\phi')^{-1} \circ (\phi' \circ f \circ \phi^{-1})$
$\circ (\phi \circ \psi^{-1})$

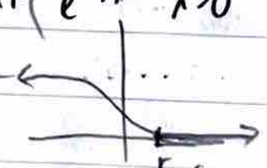
sm by def of sm. atlas (transitions)

- def: $f: M \to N$ sm. on open set U is sm @ every $p \in U$
- def: $f: M \to N$ sm. if smooth on M,
- ex: $f: M \to N$ smooth $\Leftrightarrow$ for some sm atlas $\mathcal{A}$ for M, $\mathcal{B}$ for N, $\phi \circ f \circ \psi^{-1}$ is sm. $\forall \phi \in \mathcal{B}, \psi \in \mathcal{A}$
- examples: 1) since an atlas for $\mathbb{R}^k$ is $\{id: \mathbb{R}^k \to \mathbb{R}^k\}$, $f: \mathbb{R}^n \to \mathbb{R}^m$ sm
  $\Leftrightarrow$ sm in calculus sense ($\phi = id$, $\psi = id$)
  2) $f: M \to \mathbb{R}^n$ sm $\Leftrightarrow \forall \phi: U \to V$ coord chart of M, $f \circ \phi^{-1}: V \to \mathbb{R}^n$ is sm.
- ex: 1) If $f: M \to N$ sm, then f cont.
  2) composition of sm fns is sm
- def: $C^\infty(M, N) := \{$sm maps $M \to N\}$
  $C^\infty(M) := C^\infty(M, \mathbb{R})$
- def: $f: M \to N$ is a diffeomorphism if it is a homeo & f, f^{-1} are sm.
  ↳ need both f, f^{-1} to be sm (the other does not come for free)
  e.g. $f: \mathbb{R} \to \mathbb{R}: x \mapsto x^3$
  ↳ this def is equiv to the previous one
- ex: 1) $i: S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth (inclusion)
  2) $\pi: (\mathbb{R}^{n+1} - \{\vec{0}\}) \to \mathbb{R}P^n: (x^0, \dots, x^n) \to [x^0 : \dots : x^n]$
  local coord charts for $\mathbb{R}P^n$: $U_i = \{[x^0 : \dots : x^n] : x^i \neq 0\}$ $V_i = \mathbb{R}^n$
  $\phi_i([x^0 : \dots : x^n]) = (\frac{x^0}{x_i}, \dots, \widehat{\frac{x^i}{x_i}}, \dots, x^n/x_i)$ ($\wedge$ = leave out)
  $\phi_i \circ \pi: (\mathbb{R}^{n+1} - \{\vec{0}\}) \to \mathbb{R}^n$ $(x^0, \dots, x^n) \to (x^0/x_i, \dots, \widehat{x^i/x^i}, \dots x^n/x^i)$
  is sm on $\pi^{-1}(U_i)$ ($x_i \neq 0$) $\Rightarrow \pi$ is sm
  3) $\pi \circ i$ is sm. (composition)
  4) bump function: $f: \mathbb{R} \to \mathbb{R}: x \mapsto \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} & x > 0 \end{cases}$

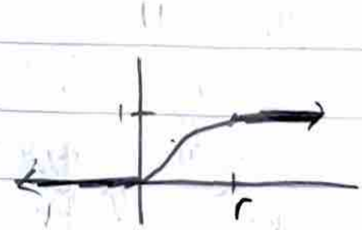


f is smooth

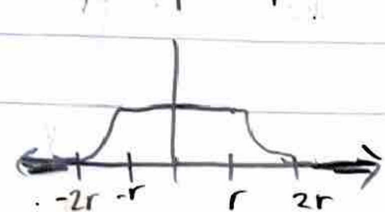
$f(r-x)$



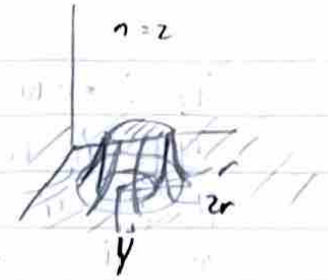
define: $\phi_r(x) := f(x) / (f(x) + f(r-x))$
↳ is sm b/c denom $\neq 0$



define $\psi_r(x) := 1 - \phi_r(|x| - r)$
↳ is smooth



let $y \in \mathbb{R}^n$, $\Psi_{r,y}(x) = \Psi_r(\|x-y\|)$
↳ (hill with flat top)



now given any $p \in M$, let $\phi: U \to V$ be a coord chart at p, say $y = \phi(p)$, $\exists r_0$ st. $B_{2r_0}(y) \subseteq V$. let $f_p: M \to \mathbb{R}: x \mapsto \begin{cases} \Psi_{y,r_0} \circ \phi(x), & x \in U \\ 0 & x \notin U \end{cases}$

" f_p is sm

2) given any open set U containing p, $\exists$ open $\mathcal{O}_p$ and $\mathcal{O}_p'$ st. $p \in \mathcal{O}_p \subseteq \mathcal{O}_p' \subseteq U$ and a sm fn $f_p: M \to \mathbb{R}$ st. $f_p = 1$ on $\mathcal{O}_p$, $f_p = 0$ outside $\mathcal{O}_p'$, f_p is smooth. f_p is called bump fn

Tangent Spaces & Linearization

- recall from calculus: directional derivative $D_v f(p) = \lim_{h \to 0} \frac{f(p+hv) - f(p)}{h}$
  ↳ for $f: M \to N$, no linear structure (how to add/subtract ??) so doesn't make sense
- goal: generalize the vector space of vectors in $\mathbb{R}^n$ based at $\vec{p}$. $(\mathbb{R}^n_p)$
- def: the derivation @ $p \in \mathbb{R}^n$ is a map $D: C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ st
  1) D is linear   2) $D(f \cdot g) = (Df) \cdot g(p) + f(p)(Dg)$ (product rule)
- Lemma: 1) for $v \in \mathbb{R}^n_p$, the directional derivative $D_v f(p)$ is a derivation @ p
  2) If $D: C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ is a derivation @ p, then $\exists \vec{v} \in \mathbb{R}^n_p$ st $D = D_v$

Pf: 1) $D_v f(p)$ satisfies linearity & product rule (calculus) ✓

2) let x^i be the coord fns $x^i: \mathbb{R}^n \to \mathbb{R}: (x^1, \dots, x^n) \mapsto x^i$, and $v^i = Dx^i$

let $f: \mathbb{R}^n \to \mathbb{R}$ be any sm. fn.

Claim: we can write $f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x-p)$ for some fns g_i, $p = (p_1, p_2 \dots)$

First note $Dc = 0$ (derivation of const $= 0$, $c \in \mathbb{R}$)

$Dc = D(c \cdot 1) \overset{\text{linear}}{=} c\,D(1 \cdot 1) \overset{\text{prod rule}}{=} c(D(1) \cdot 1 + 1 \cdot D(1)) = 2c\,D(1) \overset{\text{linear}}{=} 2D(c)$

$\Rightarrow Dc = 2Dc \Rightarrow Dc = 0$.

Then taking derivation @ $\vec{p}$ of f in Claim gives $Df = \sum_{i=1}^n v^i g_i(0)$

choosing $\vec{v} = [v^1 \dots v^n]$ gives $D_v f(p) = \sum_{i=1}^n v^i g_i(0) = Df$ so $D = D_v$ □

Pf of Claim: Fundamental thm calculus, chain rule.

- $T_p\mathbb{R}^n = \{$derivations @ $p\}$, this is a vector space
- Lemm: the map $\mathbb{R}^n_p \to T_p\mathbb{R}^n : \vec{v} \mapsto D_v$ is an isomorphism

Pf: the map is linear by vector calculus: $D_{v+aw}f = D_v f + aD_w(f)$
previous lemma says its onto; WTS it is injective. ✓ □

★ - def: let M sm. manifold, derivation @ $p \in M$ is a map $D: C^\infty(M, \mathbb{R}) \to \mathbb{R}$ s.t.
1) D is linear 2) D satisfies product rule

$T_pM = \{$derivations @ $p \in M\}$ - tangent space of M @ p

↳ tangent space is like directional derivatives

- let $f: M \to N$ sm., note if $v \in T_pM$ and $g \in C^\infty(N)$, then $g \circ f \in C^\infty(M)$
so $v(g\circ f) \in \mathbb{R}$, define: $df_p(v) : C^\infty(N) \to \mathbb{R} : g \mapsto v(g\circ f)$
Claim: $df_p(N)$ is a derivation at $f(p)$ on N.
so we have a map $df_p : T_pM \to T_{f(p)}N$.
- df_p is derivative or differential of f @ p.
  - ↳ df_p is linear
  - ↳ $f: M\to N$, $g: N \to W$ smooth then $d(g\circ f)_p = (dg_{f(p)}) \circ df_p$
  - ↳ $id: M \to M$ then $d(id)_p : T_pM \to T_pM$ is the id.
  - ↳ If $f: M \to N$ is diffeomorphism then $df_p : T_pM \to T_{f(p)}N$ is isomorphism
- thm: let $U \subseteq M$ open, $i: U \hookrightarrow M$ inclusion, induces isomorphism
$di_p : T_pU \to T_pM$ for all $p \in U$
  ↳ can compute dim of T_pM: take coord chart $\phi: U \to V$ (this is diffeo!)
  $T_pM \underset{thm}{\cong} T_pU \cong T_{\phi(p)}V \underset{thm}{\cong} T_{\phi(p)}\mathbb{R}^n \underset{Lemm}{\cong} \mathbb{R}^n_{\phi(p)} \Rightarrow M$ n-manifold, T_pM is dim n.
- lemm: If $f, g \in C^\infty(M)$ and $f = g$ on open U, then for any $p \in U$,
$v \cdot f = v \cdot g$ for all $v \in T_p(M)$. [pf by sm. bump fns]

[★ Review: In $\mathbb{R}^n$ derivation = directional derivative $T_pM = \{$derivations @ $p\}$ tangent space
U & V coord chart then $T_pM \cong T_{\phi(p)}\mathbb{R}^n$
$\dim T_p(M) = n$ if M is an n-manifold]

- If $(x^1, \dots, x^n)$ are coords in $\mathbb{R}^n$, then we get a basis
$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ for $\mathbb{R}^n_p$, so we can write $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ $(v^i \in \mathbb{R})$

★ ↳ T_pM is $\cong$ to $\mathbb{R}^n$ $\forall p$

coord charts are diffeomorphisms

- coord chart $\phi: U \to V$ at $p \in M$, let $v \in T_pM$, consider $d\phi_p(v) \in T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n_{\phi(p)}$. If $(x^1, \dots, x^n)$ coords in $\mathbb{R}^n$

  $d\phi_p(v) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$, applying inverse to both sides

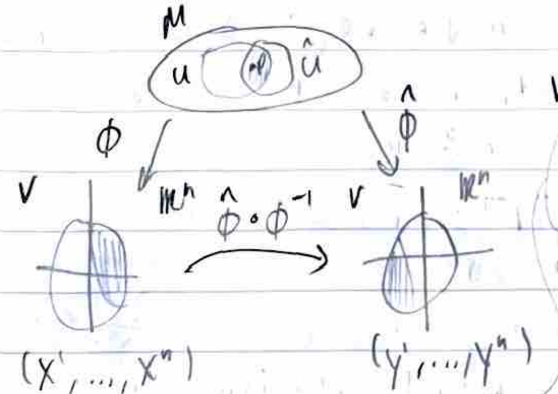
  $v = \sum_{i=1}^{n} v^i \underbrace{(d\phi^{-1})_{\phi(p)}\left(\frac{\partial}{\partial x^i}\right)}$

  ↳ abuse of notation, write as $\frac{\partial}{\partial x^i}$ b/c tangent space of M and of $\mathbb{R}^n$ are "same"

  * only makes sense with coord chart!

  $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ is a basis for T_pM (using ϕ)

what abt two diff coord charts?



$v \in T_pM$ can be written two ways:

$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$ (using ϕ)

$v = \sum_{i=1}^{n} \hat{v}^i \frac{\partial}{\partial y^i}$ (using $\hat{\phi}$)

$$(d(\hat{\phi} \circ \phi^{-1}))(d\phi_p(v)) = (d\hat{\phi}_p)(v)$$

$$(d(\hat{\phi} \circ \phi^{-1}))_{\phi(p)}\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}(\phi(p)) \frac{\partial}{\partial y^j}$$

"change basis" from $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial y^i}$

Jacobian @ $\hat{\phi} \circ \phi^{-1}$

$$\boxed{\hat{v}^j = \sum_{i=1}^{n} \frac{\partial y^j}{\partial x^i} v^i} \quad (*)$$

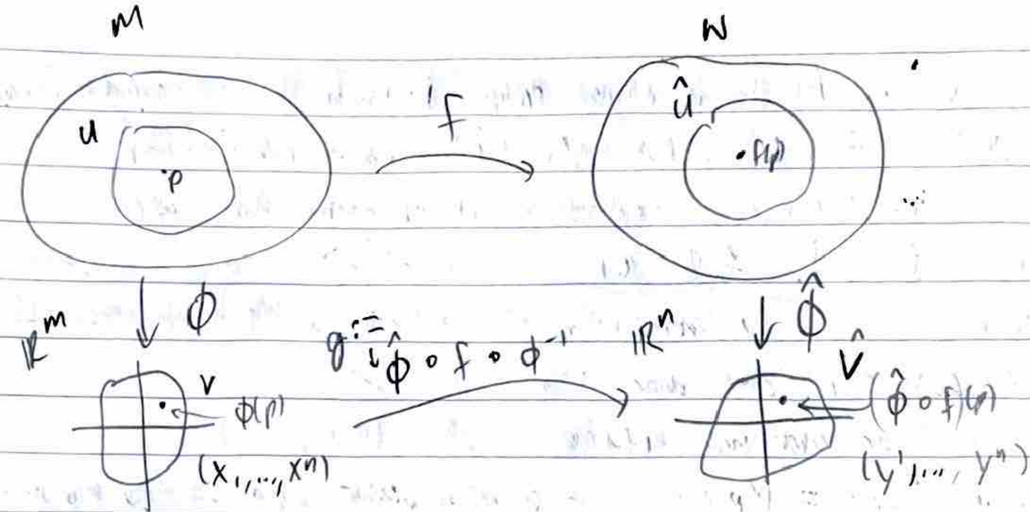
or in vector notation: $\begin{bmatrix} \hat{v}^1 \\ \vdots \\ \hat{v}^n \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \\ \cdots & \frac{\partial y^j}{\partial x^i} & \cdots \\ & \vdots & \ddots \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$

alt: a tangent vector @ $p \in M$ is an assignment of n numbers to each coord. chart containing p that transform according to $(*)$

- now: $d\phi_p$ in terms of local coords

  $f: M \to N$ smooth, $\phi: U \to V$ chart abt p in M

  $\hat{\phi}: \hat{U} \to \hat{V}$ chart abt $f(p)$ in N



If $v \in T_pM$, $df_p(v) = d\left(\hat\phi^{-1} \circ (\hat\phi \circ f \circ \phi^{-1}) \circ \phi\right)_p v$ chain rule

$= d\hat\phi^{-1}_{\hat\phi(f(p))} \circ d(\hat\phi \circ f \circ \phi^{-1})_{\phi(p)} \circ d\phi_p(v)$

$= d\hat\phi^{-1}_{\hat\phi(f(p))} \circ \underbrace{d(\hat\phi \circ f \circ \phi^{-1})_{\phi(p)}\left(\sum_{i=1} v^i \frac{\partial}{\partial x^i}\right)}_{\text{is } d\hat\phi_{f(p)}(df_p(v)) \text{ in local coords}}$

$g: \mathbb{R}^m \to \mathbb{R}^n$, $dg_p: T_p(\mathbb{R}^m) \to T_p\mathbb{R}^n$

is given by total derivative of g: $dg_p = (Dg)_p = \overset{\text{matrix}}{\left(\frac{\partial y^j}{\partial x^i}(p)\right)}$

$g(x^1,\dots,x^m) = (y^1(x^1,\dots,x^n),\dots,y^n(x^1,\dots,x^m))$

$= d\hat\phi^{-1}_{\hat\phi(f(p))} \circ (dg)_{\phi(p)}\left(\sum_i v^i \frac{\partial}{\partial x^i}\right)$

$= d\hat\phi^{-1}_{\hat\phi(f(p))}\left(\frac{\partial y^j}{\partial x^i}(\phi(p))\right)\left(\sum v_i \frac{\partial}{\partial x^i}\right)$ ← matrix multiplication

$(\dots)$ rep. by matrix $\left(\frac{\partial y^j}{\partial x^i}(\phi(p))\right)^{m,n}_{i,j}$

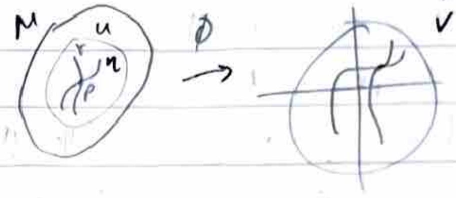
df_p generalizes total derivative from calculus

- a curve in M is a smooth map $\gamma: (a,b) \to M$, reparametrize so $0 \in (a,b)$

Consider $p = \gamma(0)$, let $P_p = \{\gamma : (a,b) \to M : \gamma(0) = p\}$

paths $\gamma, \eta \in P_p$ are equivalent if $\exists$ coord chart $\phi: U \to V$ w/ $p \in U$

s.t. $\frac{d}{dt} \phi \circ \gamma \big|_0 = \frac{d}{dt} \phi \circ \eta \big|_0$



ex) 1) true in one coord chart at p

$\iff$ true in all coord charts @ p

2) $\sim$ is an equivalence relation

define $\tilde{T}_pM = P_p / \sim$ - this is a vector space (go to local coords)

thm: $\Phi : \tilde{T}_pM \to T_pM$ by $[\gamma] \mapsto D_\gamma$ is an isomorphism

where $D_\gamma : C^\infty(M) \to \mathbb{R}$ by $f \mapsto \frac{d}{dt}(f \circ \gamma)\big|_{t=0}$

- $f: M \to N$ smooth, $p \in M$,

$df_p : \tilde{T}_pM \to \tilde{T}_pN$ by $[\gamma] \mapsto [f \circ \gamma]$

$$\begin{array}{ccc} T_pM & \to & T_pN \\ \wr\!\downarrow & & \wr\!\downarrow \\ \tilde{T}_pM & \to & \tilde{T}_pN \end{array}$$

this def agrees, i.e. this diagram commutes

☆ • <u>Linearization Thm</u>: $f : \underset{\dim m}{M} \to \underset{\dim n}{N}$ smooth. If $df_p : T_pM \to T_pN$ has maximal rank ($\operatorname{rank}(df_p) = \min\{m,n\}$) then $\exists$ coords $\phi : U \to V$ @ p and $\hat{\phi} : \hat{U} \to \hat{V}$ @ $f(p)$ such that

1) $(m \le n)$: $\hat{\phi} \circ f \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, \underbrace{0, \dots, 0}_{n-m})$ (inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$)

2) $(n \le m)$: $\hat{\phi} \circ f \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$ (projection $\mathbb{R}^m \to \mathbb{R}^n$)

• <u>Rank Thm</u>: $f : M \to N$ smooth. If $df_p : T_pM \to T_pN$ has constant rank k for all p in an open set $\mathcal{O}$, then $\forall p \in \mathcal{O}$, $\exists$ coord charts $\phi : U \to V$, $\hat{\phi} : \hat{U}, \hat{V}$ s.t. $\hat{\phi} \circ f \circ \phi(x^1, \dots, x^m) = (x^1, \dots, x^k, \underbrace{0, \dots, 0}_{n-k})$ (projection $\mathbb{R}^m \to \mathbb{R}^k$ then inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$)

☆ "$\exists$ coord charts" is "change of variables" in $\mathbb{R}^{n\,\text{or}\,m}$

• <u>Lemma</u>: let $\Psi : S \to \operatorname{Mat}(n, m; \mathbb{R})$ be continuous (for any space S) then $\operatorname{rank}\Psi : S \to \mathbb{R} : p \mapsto \operatorname{rank}(\Psi(p))$ is lower semi-continuous

(if $\operatorname{rank}(\Psi(p)) = k$ then $\exists\, \mathcal{O}$ open @ p st. $\operatorname{rank}(\Psi(x)) \ge p$ $\forall x \in \mathcal{O}$)

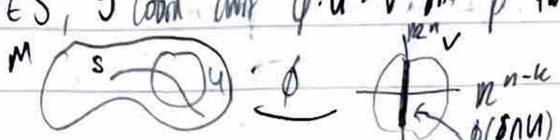
↳ df_p in local coords is a $n \times m$ matrix, depends cont. on p

↳ Rank thm $\Rightarrow$ Linearization thm

• <u>Inverse function Thm</u>: let $f : M \to N$ smooth, if $df_p : T_pM \to T_pN$ is an isomorphism, then f restricts to a diffeo on an open set containing p (it's a local diffeomorphism). So $(df^{-1})_{f(p)} = (df_p)^{-1}$.

Submanifolds

- def: an immersion is a smooth map $f: M \to N$ s.t. $df_p: T_pM \to T_{f(p)}N$ is injective $\forall p \in M$.
- def: a (smooth) embedding of $M \to N$ is a smooth $f: M \to N$ s.t.
  1) f is an immersion 2) f is a homeomorphism $M \to f(M)$ w/ subspace topology
- eg. 1) $\gamma: \mathbb{R} \to M$ is an immersion if $\gamma'(t) \neq 0$ $\forall t$
  2) $f: M \to M \times N: p \mapsto (p, q_0)$, $df_p(v) = (v, 0) \in T_pM \times T_{q_0}N = T_{(p,q_0)}(M \times N)$
  so f is an immersion (also an embedding, $M \simeq M \times \{q_0\}$)
  3) $f: \mathbb{R} \to S^1 \times S^1: t \mapsto (e^{2\pi i \alpha t}, e^{2\pi i \beta t})$ for α, β incommensurable.
  f is an injective immersion, but not an embedding
  ↳ embeddings of positive codimension cannot have dense image.
  4)  (take seq $x_n \to \infty$ in $\mathbb{R}$)
- ex: If $f: M \to N$ an injective immersion, then it is an embedding if any hold:
  1) f is an open map 2) closed map 3) M is compact
- def: let M^n be a manifold, $S \subseteq M$ is a submanifold of dim k if $\forall p \in S$, $\exists$ coord chart $\phi: U \to V$ about p for M s.t. $\phi(S \cap U) = V \cap (\{0\} \times \mathbb{R}^k)$ ↳ $\mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n$



▸ such a chart is called a slice chart

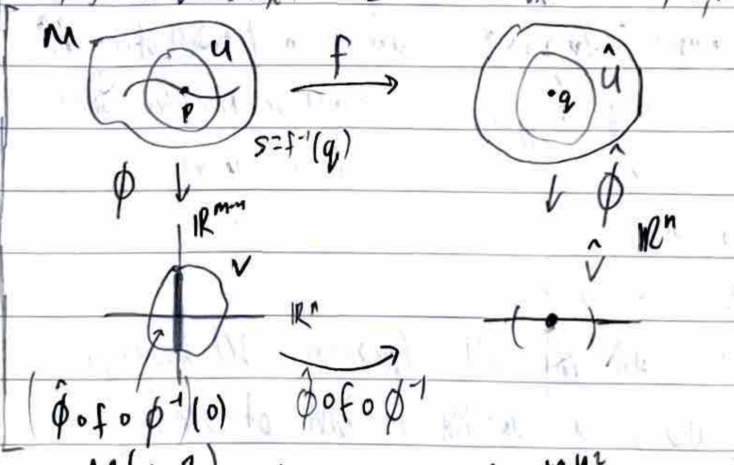
▸ $(n-k)$ is codimension of S in M

▸ M is ambient manifold of S

- ex: 1) such an S is a k-dim manifold (possibly w/ boundary)
  2) $\partial M \cap S = \partial S$ and ∂S is a submfld of ∂M of same codim.
  3) $S \subseteq M$ inherits sm str from M: restrict sm atlas to $S \cap U_\alpha$
  4) If S submfld of M and $i: S \hookrightarrow M$ is inclusion, then i is embedding
  5) If S submfld of M, $f: M \to N$ smooth, then $f|_S: S \to N$ is smooth.
- lemma: If $f: N \to M$ is a sm. embedding, then $f(N)$ is a subfld of M
  (submflds are same as images of sm embedding)
  [embedding $\Rightarrow$ rank $df_p = \dim N$ $\forall p$ $\Rightarrow$ df has maximal rank]

- def: a **submersion** is a smooth $f: M \to N$ w/ $df_p: T_pM \to T_{f(p)}N$ surjective $\forall p \in M$
  - $p \in M$ is a regular pt if df_p is surjective
  - $p \in M$ is a critical pt if df_p is not surj
  - $q \in N$ is a regular value if df_p is surj $\forall p \in f^{-1}(q)$
  - $q \in N$ is a critical value df_p is not surj "
  - f submersion $\iff$ all $p \in N$ are regular values.
- Lemma: $f: M \to N$ sm. If $q \in N$ is a regular value, then $S = f^{-1}(q)$ is a submanifold of dim $\dim M - \dim N$ (i.e., codim is dim N). Also, $T_p(S) = \ker(df_p)$
  - If $v \in T_p(f^{-1}(q))$, then $\exists$ path $\gamma: (-\varepsilon, \varepsilon) \to f^{-1}(q)$ st $v = [\gamma]$
    Then $df_p(v) = [f \circ \gamma] = 0 \in T_qN$, so $T_p(f^{-1}(q)) \subset \ker(df_p)$
- e.g. $f: \mathbb{R}^{n+1} \to \mathbb{R}: (x^1, \dots, x^{n+1}) \mapsto \sum_{i=1}^{n+1} (x^i)^2$. $df_x = [2x^1, \dots, 2x^{n+1}]$

  so if $(x^1, \dots, x^{n+1}) \neq 0$ then rank $df_x = 1$, i.e. any $a > 0$ is a regular value and $f^{-1}(a) = S^n$ is a manifold of dim n (sphere)
  $T_x S^n = \ker(df_x) = \{ v \in \mathbb{R}^{n+1} \text{ st. } [(x^1, \dots, x^n)] v = 0 \}$ (dot product/orthogonal vecs)



$\phi(p) = 0$
$\hat\phi(q) = 0$

Linearization Thm: (projection)
$\hat\phi \circ f \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$
so $\phi(S \cap U) = \{0\} \times \mathbb{R}^{m-n} \cap V$
i.e. there is a slice chart for S.

- e.g. $M(n, \mathbb{R}) = n \times n$ matrices $\cong \mathbb{R}^{n^2}$
  $GL(n, \mathbb{R})$ = invertible $n \times n$, $\mathbb{R}$ (is open subset of $M(n, \mathbb{R})$ so smooth n^2-manifold)
  $O(n) = \{ A \in GL(n, \mathbb{R}) : AA^T = I \} = \{$lin maps which preserve inner product$\}$
  $SO(n) = \{ A \in O(n) : \det A = 1 \}$ both mflds
  $U(n)$, $SU(n)$ also mflds

regular values of sm fns give submanifolds. How hard is it to find reg. vals?

- def: $A \subseteq M$ has measure 0 if $\forall \phi: U \to V$ coord charts, $\phi(A \cap U) \subseteq \mathbb{R}^n$ has measure 0.
- Sard's Thm: If $f: M \to N$ is smooth then the set of critical values in N have measure 0 (i.e. the set of regular values is dense in N)
- e.g. 1) there are cont. surjections of $[0,1] \to [0,1] \times [0,1]$ (space filling curve) but if smooth, then $\exists$ regular value $q \in I^2$ and if $p \in f^{-1}(q)$, $df_p: T_p(I) \to T_{f(p)}(I^2)$ is surjective, but there is no surjection $\mathbb{R} \to \mathbb{R}^2$. So $f^{-1}(q)$ is empty so f cannot be surjective.
  2) the image of an immersion $f: N^n \to M^m$ $(n < m)$ has ms. zero ($\operatorname{im} f \subseteq \{\text{critical values}\}$)
- ((Very) Weak) Whitney Embedding Thm: Every compact n-mfld embeds in $\mathbb{R}^N$ for some $N > n$
  1) don't need compact 2) can take $N = 2n+1$ 3) can take $N = 2n$

Pf: for each $p \in M$, let $\phi_p: U_p \to V_p$ be a coord chart abt p. $\exists$ bump fns $f_p: M \to \mathbb{R}$, $p \in \mathcal{O}_p \subseteq \mathcal{O}_p' \subseteq U_p$ open st. $f_p = 1$ on $\mathcal{O}_p$ and $f_p = 0$ outside $\mathcal{O}_p'$. Then $\{\mathcal{O}_p\}_{p \in M}$ is an open cover for M, by compactness, $\exists$ finite subcover $\{\mathcal{O}_{pi}\}_{i=1}^{\ell}$.

Note $f_{pi} \cdot \phi_{pi}: M \to \mathbb{R}^n$ by $x \mapsto \begin{cases} f_{pi}(x) \cdot \phi_{pi}(x) & x \in \mathcal{O}_{pi}' \\ 0 & \text{else} \end{cases}$

Set $\Psi: M \to \mathbb{R}^{n\ell + n}: x \mapsto (f_{p_1}(x) \cdot \phi_{p_1}(x), \dots, f_{p_\ell}(x)\phi_{p_\ell}(x), f_{p_1}(x), \dots, f_{p_\ell}(x))$.

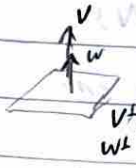
Claim: Ψ is injective. Sps $\Psi(x) = \Psi(y)$. $\exists$ some i st. $x \in \mathcal{O}_{pi}$, i.e. $f_{pi}(x) = 1$ so $f_{pi}(y) = 1$ also. Now $f_{pi}(x)\,\phi_{pi}(x) = \phi_{pi}(x) = f_{pi}(y)\,\phi_{pi}(y) = \phi_{pi}(y)$. ϕ_{pi} is a homeomorphism, hence injective, so $x = y$.

Claim: Ψ is an immersion. given $p \in M$, $\exists i$ st $p \in \mathcal{O}_{pi}$. So, $d(f_{pi} \cdot \phi_{pi})_p = d(\phi_{pi})_p$ has rank n (diffeomorphism) $d\Psi_p$ has $d(\phi_{pi})_p$ as a factor, so rank$(d\Psi_p) \geq n$, but $\dim T_pM = n$ so rank $= n$. then $d\Psi_p$ injective $\Rightarrow$ Ψ is an immersion

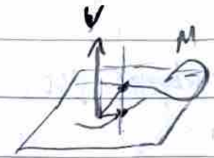
Then Ψ is an injective immersion of compact $\Rightarrow$ Ψ is an embedding. $\square$

(Weak) Whitney Embedding Thm: Any compact n-mfld embeds in $\mathbb{R}^{2n+1}$

Pf: We know M embeds in $\mathbb{R}^N$ for some N. If $v \in \mathbb{R}^N$, then $v^\perp = \{w \in \mathbb{R}^N : w \cdot v = 0\} \cong \mathbb{R}^{N-1}$. Let $\pi_v : \mathbb{R}^N \to v^\perp \cong \mathbb{R}^{N-1}$ be the orthogonal projection to $v^\perp$, which is smooth. Consider $\pi_v|_M : M \to \mathbb{R}^{N-1}$, we will show $\exists$ dense set of v s.t. $\pi_v|_M$ is an embedding, if $N > 2n+1$, so the thm will follow.



Note $\pi_v = \pi_w$ if $v\mathbb{R} = w\mathbb{R}$ (lines are same), then $[v] = [w]$ in $\mathbb{RP}^{N-1}$.
We will show $\exists$ dense set $p \in \mathbb{RP}^{N-1}$ s.t. for any $v \in$; $\pi_v|_M : M \hookrightarrow \mathbb{R}^{N-1}$ is embedding



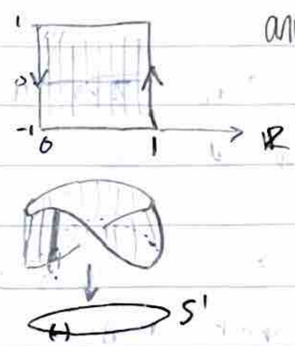
When is π_v injective? $\pi_v(x) = \pi_v(y) \iff x - y \in \mathbb{R}v$
$\iff x - y = \lambda v$ for some $\lambda \in \mathbb{R}$.
So π_v not injective iff $\exists$ distinct $x, y \in M$ s.t. $x - y = \overset{\in \mathbb{R}}{\lambda} v$

Consider $g : ((M \times M) - \Delta) \to \mathbb{RP}^{N-1} : (x,y) \mapsto [x-y]$, $\Delta = \{(x,x) : x \in M\}$ diagonal
g smooth, if p is a regular value, then $g^{-1}(p)$ is a submanifold of dim $2n - (N-1) < 2n - (2n+1-1) = 0$, i.e. $g^{-1}(p) = \emptyset$.
So no pair of pts $x \neq y \in M$ s.t. $[x-y] \neq p$. So $\forall v \in p$, $\pi_v|_M$ injective,
by Sard's thm, $\exists$ dense set of reg values $\Rightarrow$ many v s.t. $\pi_v|_M$ is injective.
Then consider when $\pi_v|_M$ is an immersion. $\Rightarrow$ have dense set of v (Sard's)
critical values have measure 0 for g, and also h (come from $\pi_v|_M$ immersion)
so union is measure 0 also, complement is dense. $\square$

Bundles

- def: a fiber bundle (twisted product, locally trivial fibration) is a quadruple (B, E, F, p) where B, E, F are topological spaces and $p: E \to B$ s.t. $\forall x \in B$, $\exists U \subseteq B$ open containing x and homeomorphism $\varphi_u : U \times F \to p^{-1}(U)$ s.t.
$$\begin{tikzcd} U \times F \arrow[r] \arrow[dr, "pr_1"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$
commutes. ($pr_1 : U \times F \to U$ is projection) F is fiber, B base space, E total space, p projection. Shorthand: $F \to E \xrightarrow{p} B$. φ_u smooth trivialization.
If F, B, E are all sm mflds, p sm map and all φ_u diffeomorphisms, then this is a smooth fiber bundle.
- ex: 1) Any F, B, let $E = B \times F$, $p : E \to B$ projection to B (any product)

2) Möbius band: $S = [0,1] \times \mathbb{R}$, $M = S/\sim$: $(t, s) \sim (-t, s+1)$



any pt in S is identified w/ some point in this rectangle.

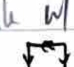
$$\begin{tikzcd} {[-1,1] \times [0,1]} \arrow[r, "q"] \arrow[d, "p(0)"] & M \arrow[d, "p"] \\ {[0,1]} \arrow[r] & S^1 = {[0,1]}/_{0 \sim 1} \end{tikzcd}$$

projection induces $p : M \to S^1$

$\bigcirc x_0$

M is fiber bundle w/ fiber $[-1,1]$ base space S^1. To see this, consider any pt. $x \in S^1 - \{x_0\}$, then $(0,1) \subset S^1$ is an open set containing x.
$[-1,1] \times (0,1) \xrightarrow{q} p^{-1}((0,1))$ is a homeomorphism, this is our local trivialization.
for x_0, let $U = (\frac{1}{2}, 1] \cup [0, \frac{1}{2}) /_{0 \sim 1} \subset S^1$
$$p^{-1}(U) = ([-1,1] \times [0, 1/2)) \cup ([-1,1] \times (1/2, 1]) / \sim$$

3) $A = [-1,1] \times S^1$ is also a bundle w/ fiber $[-1,1]$ and base S^1 (but not diffeo)

4) Klein bottle $K = [0,1]/\sim$  is a bundle w/ fibre S^1 and base S^1

5) $S^3 \subseteq \mathbb{C}^2$, $S^3 \to \mathbb{CP}^1$ by $(z_1, z_2) \mapsto [z_1 : z_2]$. $p^{-1}([z_1, z_2]) = S^1$

Claim: $S^1 \to S^3 \xrightarrow{p} S^2$ is a fiber bundle w/ fibre S^1 (Hopf fibration)

★ fiber bundles are locally like a product, but have some global 'twisting' structure

writing this gives the fiber F implicitly
↑ preimage of pts in B

- def: given bundles $\begin{array}{c}E\\ \downarrow p\\ B\end{array}$, $\begin{array}{c}E'\\ \downarrow p'\\ B'\end{array}$, a bundle map is a pair $(f, \bar{f})$ such that $\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$ commutes
- def: If $\begin{array}{c}E\\ \downarrow p\\ B\end{array}$ is a fiber bundle, then a section of the bundle is a map $\sigma : B \to E$ s.t. $p \circ \sigma = id_B$ $\begin{tikzcd} E \arrow[d, "p"'] \\ B \arrow[u, bend right, "\sigma"'] \end{tikzcd}$
  ↳ the set of all sections is denoted $\Gamma(E)$.
- def: $F \to \begin{array}{c}E\\ \downarrow p\\ B\end{array}$ f.b., local trivializations $\varphi_{u_\alpha} : U_\alpha \times F \to p^{-1}(U_\alpha)$, $\varphi_{u_\beta} : U_\beta \times F \to p^{-1}(U_\beta)$, with $p_1 \downarrow$, $\circ$, $\swarrow p$ to U_α and $p_1 \downarrow$, $\circ$, $\swarrow p$ to U_β (projection of coords → p_1)

If $U_\alpha \cap U_\beta \neq \emptyset$, can consider
$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times F \arrow[r, "\varphi_{u_\alpha}"] \arrow[dr, "p_1"'] & p^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\varphi_{u_\beta}^{-1}"] \arrow[d, "p"] & (U_\alpha \cap U_\beta) \times F \arrow[dl, "p_1"] \\ & U_\alpha \cap U_\beta & \end{tikzcd}$$

$\varphi_{u_\beta}^{-1} \circ \varphi_{u_\alpha}(x, y) = (x, (g_{\beta\alpha}(x))(y))$ where $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to \mathrm{Homeo}(F)$ continuous
this $g_{\alpha\beta}$ is called a transition function (or clutching fn)

- ex: If $U_\alpha, U_\beta, U_\gamma$ local triv, then transition fns satisfy $\boxed{g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha},\ g_{\alpha\alpha} = id_F}$ (*)
  ↳ given f.bundle and a cover of B by local trivs. $\{U_\alpha\}$, we get a collection of transition maps $\{g_{\alpha\beta}\}$ satisfying (*)
- thm: If $\{U_\alpha\}_\alpha$ is an open cover of B and $g_{\beta\alpha} : (U_\alpha \cap U_\beta) \to \mathrm{Homeo}(F)$ satisfying (*) then $E = \coprod_\alpha (U_\alpha \times F)/\sim$ where $(n, x) \in U_\alpha \times F \sim (m, y) \in U_\beta \times F$ if $n = m$ and $(g_{\beta\alpha}(n))(x) = y$ is a f.bundle over B w/ fiber F and transition fns $g_{\alpha\beta}$
  ↳ given transition maps, we can get f.bundles

—Vector Bundles—

- def: a vector bundle is a fiber bundle $\begin{array}{c}E\\ \downarrow p\\ B\end{array}$ such that $p^{-1}(x)$ is a vector space for all $x \in B$ and there are local trivializations $\varphi : U \times \mathbb{R}^n \to p^{-1}(U)$ that cover B s.t. $\varphi|_{\{x\} \times \mathbb{R}^n} : \mathbb{R}^n \to p^{-1}(x)$ is a linear isomorphism $\forall x \in U$.
- lemma 2: If $F \to \begin{array}{c}E\\ \downarrow p\\ B\end{array}$ is a fiber bundle, then it is a vector bundle iff
  1) $F = \mathbb{R}^n$ (for some n) and 2) $\exists$ cover $\{U_\alpha\}$ of B st. $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n; \mathbb{R})$ $\forall \alpha, \beta$
- ex: given sm n-mfld M, let $TM = \coprod_{x \in M} T_x M$ and $p : TM \to M$ the projection $p(v) = x \Leftrightarrow v \in T_x M$.
  1) need to give TM a topology: let $\{U_\alpha, \phi_\alpha\}$ be coord charts covering M.
  note $d\varphi_\alpha : TU_\alpha \to TV_\alpha = V_\alpha \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n : v \mapsto (d\phi_\alpha)_{p(v)}(v)$ is a 1-1 correspondence (φ_α are diffeomorphisms)

now let $\mathcal{U} = \{ d\phi_\alpha^{-1}(W)$ for all W open in $V_\alpha \times \mathbb{R}^n$ and $\alpha \in A \}$

- thm: $\mathcal{U}$ is a basis for the topology on TM and w/ this topology, TM is a $2n$-mfld. $\{ d\phi_\alpha : TU_\alpha \to TV_\alpha \}_\alpha$ give a sm atlas to TM. With this sm str, $p : TM \to M$ is a sm map, so TM is a vector bundle. Lastly, given $f : M \to N$ sm, then [ ] is a smooth bundle map

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d, "p"] & TN \arrow[d, "p"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

- def: TM is the <u>tangent bundle</u> of M.

Vector Fields and Flow

$$\begin{array}{c} TM \\ \downarrow p \;\; \uparrow v \\ M \end{array}$$

- def: a <u>vector field</u> on a manifold M is a sm section of the tangent bundle TM
  - ↳ a choice of vector at every pt of M.
- in $\mathbb{R}^n$: vector fields $v: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n : \vec{x} \mapsto (\vec{x}, F_v(x))$
  - vec fields $\iff$ functions $F_v: \mathbb{R}^n \to \mathbb{R}^n$.
- def: the set of vector fields is denoted $\mathfrak{X}(M) = \Gamma(TM)$ → set of all sections of TM
  - ↳ this is a vector space
- let $v \in \mathfrak{X}(M)$ be a v-field and a fn $f: M \to \mathbb{R}$.

★ Vector Fields act on functions

then $v \cdot f : M \to \mathbb{R}; x \mapsto v(x) \cdot f$ derivation @ x (vector in TM)
is a fn on M. so v gives a lin map $C^\infty(M) \to C^\infty(M)$
and $v \cdot (fg) = (v \cdot f)g + f(v \cdot g)$ v is a <u>derivation on M</u>

▷ vector fields $\overset{1\text{-}1}{\iff}$ derivations on M

- def: <u>Lie bracket</u>: $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

  $f, g \in C^\infty(M)$, $v, w, u \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$

  1) $[v,w] = -[w,v]$ (skew-symmetric)
  2) $[fv, gw] = fg[v,w] + f(v \cdot g)w - g(w \cdot f)v$
  3) $[av + bu, w] = a[v,w] + b[u,w]$ (linear)
  4) $[[v,u],w] + [[u,w],v] + [[w,v],u] = 0$ (Jacobi identity)
- vector space V with bilin pairing $[\cdot,\cdot]: V \times V \to \mathbb{R}$ with 1), 4) is a Lie Algebra
  - ▷ $(\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra

$$[v,w]f = v \cdot (w \cdot f) - w \cdot (v \cdot f)$$

$$[X,Y] = XY - YX$$

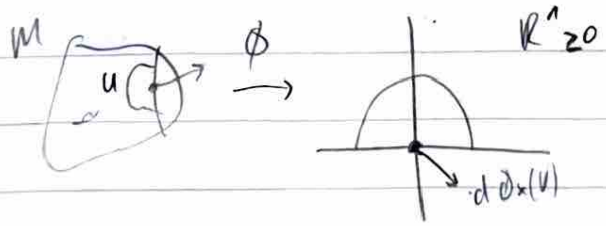
- thm: let M mfld no ∂, $v \in \mathfrak{X}(M)$ be a vector field. there are positive fns $\delta, \varepsilon : M \to (0, \infty]$ and a unique sm. fn $\Phi : W_{\varepsilon,\delta} = \{(p,t) \in M \times \mathbb{R} : -\varepsilon(p) < t, \delta(p)\} \to M$ st
1) $\Phi(p,0) = p$ 2) $d\Phi_{(p,t)}\left(\frac{\partial}{\partial t}\right) = v(\Phi(p,t))$ also satisfies $\Phi(\Phi(p,s),t) = \Phi(p,s+t)$
If v has compact support, then $\varepsilon, \delta = \infty$ i.e. $\Phi : M \times \mathbb{R} \to M$

- Fundamental thm of ODE's: $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^n$ smooth, for $a \in \mathbb{R}$, $x_0 \in U$
Existence: $\exists$ open interval J_0 containing a, an open set $U_0 \subseteq U$ containing x_0 st $\forall p \in U_0$, $\exists \gamma_p : J_0 \to U$ st. $(*)$ $[\gamma_p'(t) = F(\gamma(t))$ and $\gamma_p(a) = p]$
Uniqueness: If $\gamma, \tilde{\gamma}$ both satisfy $(*)$ then $\gamma(t) = \tilde{\gamma}(t)$ on their common domain
Smoothness: If J_0, U_0 as above the map $\Gamma : U_0 \times J_0 \to U : (p,t) \mapsto \gamma_p(t)$ is smooth

1) $\gamma^p(t) = \Phi(p,t)$, then $\gamma^p : (-\varepsilon(p), \delta(p)) \to M$ is a curve st. $\gamma^p(0) = p$ and $(\gamma^p)'(t) = (d\gamma^p)_{\gamma(t)}\left(\frac{\partial}{\partial t}\right) = v(\gamma(t))$
this is the flow line or integral curve of v through p

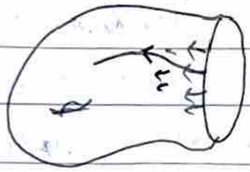
2) any map $\Phi : W_{\delta,\varepsilon} \to M$ satisfying $\Phi(p,0) = p$, $\Phi(\Phi(p,s),t) = \Phi(p,s+t)$ is called a flow on M or a dynamical system on M
given a flow Φ, set $v(p) = d\Phi_{(p,0)}\left(\frac{\partial}{\partial t}\right)$, $v : M \to TM$ is a sm. section
this gives a vector field associated to Φ called velocity field of Φ
3) fix $t \in (-\min \varepsilon, \min \delta)$ and set $\phi^t : M \to M : p \mapsto \Phi(p,t)$
this is a sm map and if $t, -t \in (\min \varepsilon, \min \delta)$ then
$\phi^t \circ \phi^{-t} = \Phi(\Phi(p,-t),t) = \Phi(p,0) = p$
so also, $\phi^{-t} \circ \phi^t = \mathrm{id}_M = \phi^t \circ \phi^{-t}$
so ϕ^t is a diffeo: $\boxed{\text{good way to build diffeos!}}$

- so If M mfld w/ ∂, then we say $v \in T_xM$, $x \in \partial M$ points out of M if in some local chart at x, $\phi : U \to V$, $d\phi_x(v)$ has negative x-coord.



- points out M ~~every~~ every coord chart

- ex: If v is a vector field on M and $v(\partial)$ never points out of M then $\exists \delta : M \to (0,\infty)$ s.t. $\Phi : W_{0,\delta} \to M$ as in thm 1 is well defined (and if v has compt supp then $\Phi : M \times [0,\infty) \to M$)



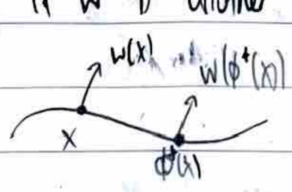
- using flows, $\exists$ diffeo sending any 1 pt to another pt
- thm: every open nbhd of ∂M in a cmpt mfld M contains a collar nbhd
i.e. $\exists$ a sm map $\Psi : (\partial M \times [0,\varepsilon)) \to M$ s.t. $\Psi|_{\partial M \times \{0\}} : \partial M \times \{0\} \to \partial M$ is the id map and $\Psi : \partial M \times [0,\varepsilon) \to \operatorname{Im}\Psi$ is a diffeomorphism

— Lie derivative —

$v \in \mathfrak{X}(M)$, $\Phi : W_{\varepsilon,\delta} \to M$ be its flow, $\phi^t : M \to M$ is associated diffeo for small t ($\phi^t(x) = \phi(x,t)$)

given $f : M \to \mathbb{R}$, the Lie derivative of f along v is
$$\mathcal{L}_v f(x) = \lim_{t\to 0} \left(f \circ \phi(x,t) - f(x)\right)/t = \frac{d}{dt}\left(f \circ \Phi(x,t)\right)\Big|_{t=0}$$
★ (rate of change of f along flow line) ★

If w is another vector field, the Lie deriv of w along v is



$$\mathcal{L}_v w = \lim_{t\to 0} \left(d\phi^{-t}_{\phi^t(x)}(w(\phi^t(x))) - w(x)\right)/t$$
$$= \frac{d}{dt}\left(d\Phi^{-t}_{\phi^t(x)}(w(\phi^t(x)))\right)\Big|_{t=0}$$

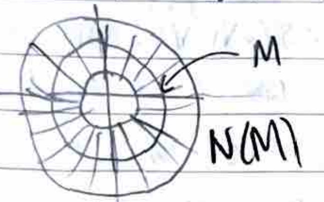
- thm: 1) $\mathcal{L}_v f = v \cdot f = df(v)$ all these derivatives are the same
2) $\mathcal{L}_v w = [v,w]$
↳ vector fields are same if they act on functions in the same way

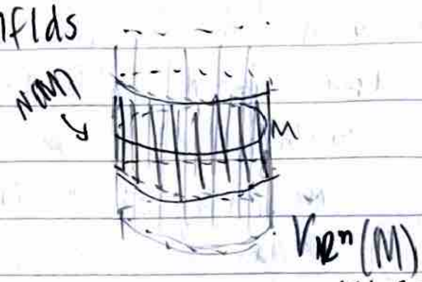
— end exam 1 content —

Approximations and Stability

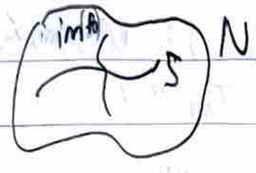
- <u>thm</u>: let M sm. mfld, $f: M \to \mathbb{R}^k$ contin. Given any cont pos fn $\delta: M \to \mathbb{R}^{>0}$, $\exists\ \hat{f}: M \to \mathbb{R}^k$ smooth such that $\|f(x) - \hat{f}(x)\| < \delta(x)\ \forall x \in M$
and if f is sm on a closed $A \subseteq M$ then we can take $\hat{f} = f$ on A
↳ any cont fn can be "jiggled" to be a smooth fn
↳ "smooth fns are dense in cont. fns"
- <u>def</u>: given an open cover $\{U_\alpha\}_{\alpha \in A}$ for a mfld M, a <u>partition of unity</u> subordinate to $\{U_\alpha\}$ is a collection of fns $\{\psi_\alpha : M \to \mathbb{R}^k\}$ such that
1) $0 \le \psi_\alpha(x) \le 1$ 2) support of $\psi_\alpha \subseteq U_\alpha$ 3) $\forall x$, $\exists$ nbhd V_x st. $\psi_\alpha|_{V_x} \neq 0$ for only finitely many α 4) $\sum_{\alpha \in A} \psi_\alpha(x) = 1\ \ \forall x \in M$
- lemma: Any open cover of a sm. mfld has a partition of unity subordinate to it.
- Cor: let M sm mfld, $A \subseteq M$ <u>closed</u>, $f: A \to \mathbb{R}^k$ smooth. For any $U \subseteq M$ open w/ $A \subseteq U$, $\exists$ sm fn $\tilde{f}: M \to \mathbb{R}^k$ st. $\tilde{f} = f$ on A and supp $\tilde{f} \subseteq U$
↳ can extend sm fn on closed subset to sm on whole mfld.
- Vector bundle $E \xrightarrow{p} M$, subbundle $G \subseteq E$, $G \downarrow p|_G$ over M, fiber dim of $E = n$, fib dim of $G = k$
set $E/G = \bigcup_{x \in M} E_x / G_x$ (E_x = fiber of E above x, $p^{-1}(x)$)
let $q: E/G \to M$ be the obvious projection
let $U \times \mathbb{R}^n \xrightarrow{\phi} p^{-1}(U)$ be a local trivialization of E
$p_1 \downarrow\ U \swarrow p$ for each $x \in U$, $\phi^{-1}(G_x)$ is a k-dim subspace of $\mathbb{R}^n$
and $\exists$ matrix A_x st. $A_x(\mathbb{R}^k \times \{0\}) = \phi^{-1}(G_x)$
E/G is a $(n-k)$ vector bundle
- <u>ex</u>: if S is a subfld of M then TS is a subbundle of $TM|_S$
- <u>def</u>: the normal bundle of S in M is $\nu_M(S) = TM|_S / TS$
↳ for $\mathbb{R}^n$: if M subfld of $\mathbb{R}^n$, let $TM^\perp := \{v \in T_x\mathbb{R}^n \mid x \in M,\ v \perp T_x M\}$
at each $x \in M$, $T_x\mathbb{R}^n = T_x M \oplus (TM^\perp)_x$ and $\nu_{\mathbb{R}^n}(M) \cong TM^\perp$
- <u>thm</u>: If M^m compact submfld of $\mathbb{R}^n$ then in any open nbhd U of M, $\exists$ open set $N(M)$ containing M st. $N(M)$ is an open disk bundle over M: $B^{n-m} \to N(M)$ and $\exists$ a subdisk bundle $N' \subseteq \nu_{\mathbb{R}^n}(M)$ st. $N' \cong N(M)$.
$\downarrow q$ over M
In particular, $q: N(M) \to M$ is the id on M and q submersion
▸ true for M not compact, & for any subfld S of M

* normal bundles are mflds





- def: $f_0, f_1 : X \to Y$ are homotopic if $\exists$ cont $F: X \times [0,1] \to Y$ s.t. $f(x,0) = f_0(x)$, $f(x,1) = f_1(x)$ $\forall x, X$. F is a homotopy.
  ↳ if X, Y, F sm then call it a sm homotopy
- thm: N, M sm mflds, M no boundary, $f: N \to M$ cont. Then f is homotopic to a sm. map
  If f is sm on closed A, then homotopy can be id on A
- Cor: $f_0, f_1 : N \to M$ sm, f_0 homotopic to f_1 $\Leftrightarrow$ sm, homotopic
- def: $f_0 : M \to N$ is stable if for any sm. homotopy f_t of f_0, $\exists \varepsilon > 0$ st. f_t also has this property for all $t < \varepsilon$.
- thm: the following props. of sm maps from $M \to N$ (both compact) stable:
  1) local diffeo
  2) immersions
  3) submersions
  4) maps transverse to a fixed closed sfld $S \subseteq N$
  5) embeddings
  6) diffeomorphisms
- def: $f: M \to N$ is transverse to $S \subseteq N$ if $\forall p \in f^{-1}(S)$ we have $T_{f(p)} N$ is spanned by $\mathrm{im}(df_p)$ and $T_{f(p)} S$ and if f transverse to S then $f^{-1}(S)$ is a mfld of codim = codim S in N

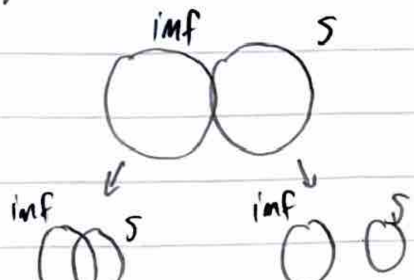


Transversality

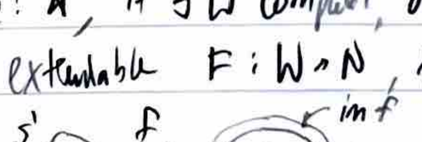
- $f \pitchfork S \to f$ transverse to S
- ex: $\partial M \neq \emptyset$, $f|_{\partial M} : \partial M \to N$ is $\pitchfork$ to S and $f: M \to N$ is $\pitchfork$ to S then $f^{-1}(S)$ is a submfld of M, $\partial f^{-1}(S) \subset \partial M$.
- thm: let M, N, X be sm mflds, $S \subset N$ submfld. If $F: M \times X \to N$ transverse to S, ($F|_{\partial M \times X} \pitchfork S$ if has ∂), then except on a set of ms 0, $f_x : M \to N : p \mapsto F(p,x)$ is $\pitchfork$ to S.
  - ↳ many maps are transverse
  - ↳ Pf: prove regular values satisfy, then Sard's thm
- thm 2: let M, N sm mflds (M possibly w/ ∂), $S \subseteq N$ submfld. For any sm $f: M \to N$, there is a homotopy to a sm map $\hat{f}: M \to N$ st. $\hat{f} \pitchfork S$ and $\hat{f}|_{\partial M} \pitchfork S$.
- thm 3: (same conditions ↑). If $\exists$ a closed set C of M where $f \pitchfork S$, ($f|_{\partial M} \pitchfork S$ on C if has ∂) then f sm homotopic to $g: M \to N$ st. $g \pitchfork S$ ($g|_{\partial M} \pitchfork S$) and $g = f$ on open nbhd of C.
- thm: every compact conn. 1-mfld is diffeo to $[0,1]$ or S^1
- thm 4: let M sm, compact mfld w/ ∂. then is no cont retraction of M to ∂M (no map $f: M \to \partial M$ st $f = id$ on ∂M).
  - ↳ Cor 5 (Brower's fixed pt): any cont map $D^n \to D^n$ has a fixed pt

Mod2 intersection —

given $f: M \to N$ sm can homotope to transverse f_1 (thm 2)



- $I_2(f,S) = \# f_1^{-1}(S) \mod 2$
- thm: I_2 is well def ($f_1 \sim f_2$, $f_1 \pitchfork f_2$ $\Rightarrow I_2(f_1, S) = I_2(f_2, S)$)
- thm: ★, if $\exists W$ compact, $\partial W = M$, $f: M \to N$ extendable $F: W \to N$, then $I_2(f,S) = 0$



$f \pitchfork S$, $f^{-1}(S) = \{pt\}$ $\Rightarrow I_2(f, S) = 1$.

$f: S^1 \to S^2$ can be extend $F: D^2 \to S^2$

$\Rightarrow S^2$ not diffeo to T_2

★ [M, N mflds, $S \subset N$ submfld, $\dim M + \dim S = \dim N$, M, S no ∂, S closed in N, M compact]

(*) • def: $f: M \to N$, $S \subseteq N$ (submfd), $\dim S + \dim M = \dim N$, S closed, M compact, M, S no boundary

$I_2(f, S) = $ (# pts in $f_1^{-1}(S)$ where $f_1 \pitchfork S$, f_1 homotopic to f) mod 2

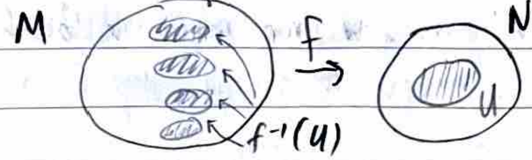
• def: degree mod 2 of a map $f: M \to N$, M compact, N connected, is $\deg_2 f = I_2(f, \{p\})$ ($p \in N$)

• thm: for any $p_1, p_2 \in N$, $I_2(f, \{p_1\}) = I_2(f, \{p_2\})$ ($\deg_2$ is well defined)

▷ Cor: homotopic maps have same degree mod 2

need ▷ Lemma: for any regular value p of f, $\exists$ open nbhd U of p s.t. $f^{-1}(U) = U_1 \cup \ldots \cup U_k$ where $U_1, \ldots, U_k$ disj. and $f: U_i \to U$ diffeo

(stack of pancakes / coins thm)



• thm: Sps M, N, S, $f: M \to N$ (all as above *) if $\exists$ a compact mfld W s.t. $\partial W = M$, and f can be extended to $F: W \to N$ then $\deg_2(f) = 0$

• thm: every complex poly of odd deg has a root

Cotangent Bundles and 1-forms

Review: V vector space, the dual space of V is $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$
$= \{\mathbb{R}\text{-linear maps } L: V \to \mathbb{R}\}$

If $e_1, \dots, e_n$ basis for V, $e^i: V \to \mathbb{R} : e_j \mapsto \delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

- $e^1, \dots, e^n$ form a basis for V^*, called dual basis to $e_1, \dots, e_n$
- $\dim V = \dim V^*$
- let $a \in V^*$, $v \in V$, write $a = \sum_{i=1}^n a_i e^i = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $v = \sum_{i=1}^n v^i e_i = [v^1 \dots v^n]$
  $a(v) = \sum_{i=1}^n a_i v^i$ $\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot [v^1 \dots v^n] \text{ dot prod}\right)$
- $a_i = a(e_i)$

If $L: V \to W$ linear map, $L^*: W^* \to V^*$ linear map induced by $\underset{\in W^*}{a} \mapsto a \circ L \in V^*$

- $(T \circ S)^* = S^* \circ T^*$
- $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$
- $e_1, \dots, e_n$ basis for V, $f_1, \dots, f_m$ basis for W then $L(e_i) = \sum_{j=1}^m L_i^j f_j$
  (the matrix L_i^j reps. L in these bases). $L^* = L^T$.

- def: let M mfld, the cotangent space of M at p is $T_p^*M := (T_pM)^* = \mathrm{Hom}(T_pM, \mathbb{R})$
  - if $f: M \to \mathbb{R}$, then $df_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R} : v \mapsto df_p(v) = v \cdot f$ so $df_p \in T_p^*M$
- local coords: let $\varphi: U \to V$ coord chart abt p, $q \mapsto (x^1(q), \dots, x^n(q))$.
  $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ is a basis for T_pM. let $dx^1, \dots, dx^n$ be dual basis $\left(dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j\right)$
  do same for $\hat{\varphi}: \hat{U} \to \hat{V}$ @ $q = f(p)$ $(f: M \to N$ sm$)$
  $df_p : T_pM \to T_qN$ local coords is matrix $\left(\frac{\partial f^j}{\partial x^i}\right)$
  $df_p^* : T_q^*N \to T_p^*M : (d)f_p^*(a) = a \circ df_p$ (pull back of a by f)
  usually omit d because its clear you're talking about dualizing w/ $*$
  - $(f \circ g)^*_{f(g(p))} = g^*_{f(x)} \circ f^*_{f(g(p))}$
  - $(\mathrm{id}_M)^*_p = \mathrm{id}_{T_p(M)}$
  - in local coords f^* is $\left(\frac{\partial f^j}{\partial x^i}\right)^T$ (transpose)

$\to$ gradient transforms as a covector under coordinate change, not as a vector

- def: $T^*M = \bigsqcup_{p \in M} T_p^*M$ is the cotangent space of M.

topology and bundle structure: let $\varphi: U \to V$ be coord chart for M,
note $TV = V \times \mathbb{R}^n$ and $T^*V = V \times \mathbb{R}^n$ and so $\varphi^*: V \times \mathbb{R}^n \to T^*U = T^*M|_U$ is
a bijection $\Rightarrow$ these are coord charts for T^*M.

- def: a section of T^*M is a 1-form (or covector field) $\begin{array}{c} T^*M \\ \pi\downarrow \circlearrowright \alpha \\ M \end{array}$
- def: the space of sections is $\Omega^1(M) = \Gamma(T^*M)$
- note: given $\alpha \in \Omega^1(M)$, this gives a linear map $\mathfrak{X}(M) \xrightarrow{\varphi_\alpha} C^\infty(M) : V \mapsto \alpha(V)$
- ex: $\Omega^1(M) \to \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M))$, $\alpha \mapsto \varphi_\alpha$ is an $\cong$
- def: $d: C^\infty(M) \to \Omega^1(M) : f \mapsto df$ is the exterior derivative
- 1) $d(af+bg) = adf + bdg$
  2) $d(fg) = fdg + gdf$
  3) If $\varphi: M \to N$ sm, define $\varphi^*: C^\infty(N) \to C^\infty(M)$ by $f \mapsto f \circ \varphi$, then diagram commutes.
  $$\begin{array}{ccc} C^\infty(N) & \xrightarrow{\varphi^*} & C^\infty(M) \\ d\downarrow & \circ & \downarrow d \\ \Omega^1(N) & \xrightarrow{\varphi^*} & \Omega^1(M) \end{array}$$
  4) $f \in C^\infty(N)$, $\alpha \in \Omega^1(N)$ then $\varphi^*(f\alpha) = (\varphi^* f)(\varphi^*\alpha)$
- eg: $f(x,y,z) = (x^2y, y\sin z)$ $f: \mathbb{R}^3 \to \mathbb{R}^2$ w/ $u = x^2y$, $v = y\sin z$,
  $\alpha(u,v) = u^2 dv + v du \in \Omega^1(\mathbb{R}^2)$. $\alpha: \mathbb{R}^2 \to T^*(\mathbb{R}^2) = \bigsqcup_p \mathrm{Hom}(T_p(\mathbb{R}^2), \mathbb{R})$
  now: $f_p^*: T^*_{f(p)}\mathbb{R}^2 \to T^*_p \mathbb{R}^3$ $f^*(du) = d(u \circ f) = d(x^2y) = 2xy\,dx + x^2 dy$
- a 1-form on $[a,b] \subseteq \mathbb{R}$ can be written $\alpha = f(t)dt$ for some $f: [a,b] \to \mathbb{R}$, t is coord on $[a,b]$.
- def: the integral of α on $[a,b]$ is $\int_{[a,b]} \alpha = \int_a^b f(t)dt$
- $\alpha \in \Omega^1(M)$, $C \subseteq M$ compact 1-mfld w/ a direction, can parametrize C by some map $\gamma: [a,b] \to C \subseteq M$ so as t increases, C is traversed in chosen direction def: $\int_C \alpha = \int_{[a,b]} \gamma^* \alpha$. (like line integral)
- lemma: $\int_C df = f(B) - f(A)$, C path $A \to B$.
- def: If $\alpha = df$ for some $f \in C^\infty(M)$ α is called exact
- thm: $\alpha \in \Omega^1(M)$, TFAE:
  1) α is exact
  2) $\int_C \alpha = 0$ $\forall$ loops $C \subset M$
  3) $\int_C \alpha$ only depends on the endpts of C

Tensors

- let V, W vec. spaces.

  let $F(V \times W)$ be the vec. space generated by $V \times W$ ← finite linr combs

  let $R(V \times W)$ be the subspace of $F(V \times W)$ generated by

  $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$

  $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$

  $(av, w) - a(v, w)$

  $(v, aw) - a(v, w)$

- def: the tensor product of V and W is $V \otimes W = F(V \times W) / R(V \times W)$

  the coset of (v, w) is denoted $v \otimes w$

- rmk: 1) left and right distributive 2) bilinear ← tensor is not commutative

- Universal Property: $\varphi : V \times W \to V \otimes W : V \times W \xrightarrow{i} F(V \times W) \xrightarrow{p} V \otimes W$

  If U is a vector space and $\ell : V \times W \to U$ is bilinear then $\exists$ unique $\tilde{\ell} : V \otimes W \to U$ st. $\tilde{\ell} \circ \varphi = \ell$

$$\begin{tikzcd} V \times W \arrow[r, "\ell"] \arrow[d, "\varphi"'] & U \\ V \otimes W \arrow[ur, "\tilde{\ell}"'] & \end{tikzcd}$$

  $\tilde{\ell}$ linear!

- ex: 1) If X and $\varphi : V \times W \to X$ satisfy universal prop then $X \cong V \otimes W$

  2) $\{\text{bilinear maps } V \times W \to U\} \overset{1-1}{\longleftrightarrow} \{\text{linear maps } V \otimes W \to U\}$

  3) $V \otimes W \cong W \otimes V$

  4) $V \otimes (W \otimes U) \cong (V \otimes W) \otimes U$

  5) $e_1, \ldots e_n$ basis for V, $f_1, \ldots, f_m$ basis for W then $e_i \otimes f_j$ basis for $V \otimes W$

  So $\dim(V \otimes W) = \dim V \cdot \dim W$

- lemma: $V^* \otimes W^* \cong (V \otimes W)^* = \mathrm{Bilin}(V \times W, \mathbb{R})$

☆ - Cor: $V \otimes W \cong V^{**} \otimes W^{**} \cong (V^* \otimes W^*)^* \cong \mathrm{Bilin}(V^* \times W^*, \mathbb{R})$

- ex: $V^* \otimes W \cong \mathrm{Hom}(V, W)$ by $(\alpha, w) \mapsto (\ell_{\alpha, w} : V \to W : v \mapsto \alpha(v) w)$

  ☆ elts in $V \otimes W$ do not look like just $v \otimes w$, need all finite sums of these.

- notation: $T^p_q(V) = \underbrace{V \otimes \ldots \otimes V}_{q \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{p \text{ times}} = \mathrm{MultiLin}(V^* \times \ldots \times V^* \times V \times \ldots \times V, \mathbb{R})$

  $T^0_0(V) = \mathbb{R}$

- def: Set $T_*(V) = \sum_k T^0_k(V)$. this is the tensor algebra of V

  ↳ $T_*(V)$ is a vector space with multiplication

  ↳ $a \in T^0_k(V)$, $b \in T^0_\ell(V) \to a \otimes b \in T^0_{k+\ell}(V)$

- def: $T^k(V) = T^k_0(V) = V^* \otimes \dots \otimes V^* = \text{Multilin}(V \times \dots \times V, \mathbb{R})$
- def: for a mfld M, $T^k(M) = \bigsqcup_{p \in M} T^k(T_pM)$

$$T_k(M) = \bigsqcup_{p \in M} T_k(T_pM)$$

$$T^\ell_k(M) = \bigsqcup_{p \in M} T^\ell_k(T_pM)$$

  - ▷ these are all vector bundles over M w/ fibers having dimension $n^{k+\ell}$ ($\dim M = n$)
  - ▷ $T^1(M) = T^*(M)$ (cotangent space)
  - ▷ $T_1(M) = T(M)$ (tangent space)
  - ▷ $T^0M = M \times \mathbb{R} = T_0M$
- given $\sigma \in \Gamma(T^kM)$, it can be written as $\{dx^i\}_{i=1}^n$ form a basis for T^*M

  $\sigma = \sum_{i_1, \dots, i_k = 1}^{n} \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$

  given $f: M \to N$, then $f^*: T^*N \to T^*M$ so also get

  $f^*: T^kN \to T^kM$ and $f^*\sigma \in \Gamma(T^kM)$
- ex: $f: M \to N$, $g: N \to \mathbb{R}$
  1) $f^*(g\sigma) = (f^*g)(f^*\sigma) = (g \circ f) f^*\sigma$
  2) $f^*(\sigma \otimes \tau) = (f^*\sigma) \otimes (f^*\tau)$
  3) $f^*: \Gamma(T^kN) \to \Gamma(T^kM)$ is linear
  4) $h: N \to W$, $(h \circ f)^* = f^* \circ h^*$
  5) $(id_N)^* = id_{\Gamma(T^kN)}$
- for $\mathbb{R}^n$ (i.e. in local coords for any mfld)

  $f: \mathbb{R}^n \to \mathbb{R}^m: (x^1, \dots, x^n) \mapsto (y^1(x^1, \dots x^n), \dots, y^m(x^1, \dots, x^n))$

  $\sigma = \sum \sigma_{i_1 \dots i_k} dy^{i_1} \otimes \dots \otimes dy^{i_k} \in \Gamma(T^k\mathbb{R}^m)$

  then $f^*(dy^{i_1}) = d(f^*y^{i_1}) = d(y^{i_1} \circ f) = dy^{i_1}(x^1, \dots, x^n)$ (do for each coord)

Forms

- def: $\varphi \in V^* \otimes \dots \otimes V^* = \text{Multilin}(\overbrace{V \times \dots \times V}^{k}, \mathbb{R})$ is <u>alternating</u> if $\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma)\, \varphi(v_1, \dots, v_k)$ for any $v_i \in V$, $\sigma \in S_k = \text{Sym}(k)$
- def: $\Lambda^k(V) = \{\varphi \in \text{Multilin}(\overbrace{V \times \dots V}^{k}, \mathbb{R}) : \varphi \text{ is alternating}\} \subset V^* \otimes \dots \otimes V^*$
- lemma: $\varphi \in \text{Multilin}(V \times \dots V, \mathbb{R})$. TFAE:
  1) φ is alternating $(\varphi \in \Lambda^k(V))$
  2) $\varphi(v_1, \dots, v_i, \dots, v_i, \dots, \dots, v_k) = 0$ (plugging in same vec twice gives 0)
  3) $\varphi(v_1, \dots, v_k) = 0$ when $\{v_1, \dots, v_k\}$ is not lin indep. (is lin. dep)
- def: $\text{Alt}: \text{Multilin}(V \times \dots \times V, \mathbb{R}) \to \Lambda^k(V) : \varphi \mapsto \frac{1}{k!} \sum_{\sigma \in S_n} (\text{sgn}\, \sigma) \varphi^\sigma$ (where $\varphi^\sigma(v_1, \dots, v_k) = \sigma(v_{\sigma(1)}, \dots, v_{\sigma(k)})$)
- ex: $\varphi \in V^* \otimes V^*$, $\text{Alt}(\varphi)(v_1, v_2) = \frac{1}{2}(\varphi(v_1, v_2) - \varphi(v_2, v_1))$
- 1) Alt is a linear map
  2) for any $\varphi \in T^k(V) = \text{Multilin}(V \times \dots \times V, \mathbb{R})$ $\text{Alt}(\varphi)$ is alternating
  3) If $\varphi \in \Lambda^k(V)$ then $\text{Alt}(\varphi) = \varphi$
  4 $\text{Alt}(\text{Alt}) = \text{Alt}$ (like a projection)
- def: let $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$. the <u>wedge product</u> of ω and η is $\omega \wedge \eta = ((k+l)!/(k!\,l!))\, \text{Alt}(\omega \otimes \eta) \in \Lambda^{k+l}(V)$
- ex: $\omega, \eta \in \Lambda^1(V) = V^*$, $\omega \wedge \eta = \frac{(1+1)!}{1!\,1!} \text{Alt}(\omega \otimes \eta) = \frac{2!}{2}(\omega \otimes \eta - \eta \otimes \omega) = \omega \otimes \eta - \eta \otimes \omega$
- ex: $e_1, \dots e_n$ basis for V, then $e^1, \dots, e^n$ dual basis for V^*, then $\{e^{i_1} \otimes \dots \otimes e^{i_k}\}_{1 \le i_j \le k}$ basis for $V^* \otimes \dots \otimes V^*$
  1) $\{e^{i_1} \wedge \dots \wedge e^{i_k}\}_{1 \le i_1 < i_2 < \dots < i_k \le n}$ is a basis for $\Lambda^k(V)$
  2) $\dim(\Lambda^k(V)) = \binom{n}{k}$ $\quad(\Lambda^k(V) = 0$ if $k < 0$ or $k > n)$
- def: $\Lambda(V) = \Lambda^0(V) \oplus \dots \oplus \Lambda^n(V)$ $(\dim V = n)$ is <u>exterior algebra</u> of V
- lemma: 1) $\wedge$ is linear (pull out const, distribute over sums) 2) associative
  3) $\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega$ $\quad(\omega \in \Lambda^k(V), \eta \in \Lambda^l(V))$
  4) $(\omega^1 \wedge \dots \wedge \omega^k)(v_1, \dots, v_k) = \det(\omega^j(v_i))$, $\omega^i \in V^* = \Lambda^1(V)$, $v_j \in V$
- cor: $L: V^n \to W^n$ linear. $e_1, \dots, e_n$ basis for V, $f_1, \dots, f_n$ basis for W. Express L as matrix $M = (m^j_i)$ in this basis. then $L^*(f^1 \wedge \dots \wedge f^n) = (\det M)\, e^1 \wedge \dots \wedge e^n$

wedge product is "just like" adding vectors to make subspaces and $=0$ iff vectors linearly dependent

- def. M mfld, $\Lambda^k(M) = \bigcup_{p\in M} \Lambda^k(T_pM)$
  ↳ $\Lambda^k M$ is a mfld and vector bundle over M, fibres $\Lambda^k T_pM \subseteq T_k(T_p^* M)$
- def: $\Omega^k(M) = \Gamma(\Lambda^k(M))$ (sections of bundle)
  $\alpha \in \Omega^k(M)$ is a <u>k-form</u>
- note: 1) $\Lambda^1 M = T^* M$
  2) $\Lambda^0 M = M \times \mathbb{R}$, $\Omega^0(M) = C^\infty(M)$
  3) $f: M \to N$ induces $f^*: T^*N \to T^*M$ induces $f^* \Lambda^k N \to \Lambda^k M$
  induces $f^*: \Omega^k(N) \to \Omega^k(M)$
- <u>lemma</u>: $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$
  ↳ $f^*(\omega \otimes \eta) = f^*\omega \otimes f^*\eta$
- <u>thm</u>: $f: M^n \to N^n$, $\varphi: U \to V$, $\varphi': U' \to V'$ for M, N w/ $(x^1, \dots, x^n)$, $(y^1, \dots, y^n)$ local st $f(U) \subseteq U'$. Set $F := \varphi' \circ f \circ \varphi^{-1}$, $F^*(dy^1 \wedge \dots \wedge dy^n) = \det(DF)\, dx^1 \wedge \dots \wedge dx^n$

- thm: M dim n $\exists!$ map $d: \Omega^k(M) \to \Omega^{k+1}(M)$ for $k = 0, \dots, n$ st
  1) $d(a\alpha + b\beta) = a\,d\alpha + b\,d\beta$ $a,b \in \mathbb{R}$
  ★ → 2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$ $(\alpha \in \Omega^{|\alpha|}(M))$
  3) $d^2 = 0$
  4) df is the exterior derivative if $f \in C^\infty(M) = \Omega^0(M)$
     ↳ for a 0-form ($C^\infty(M)$), df is the exterior derivative, i.e., the unique 1-form st $\forall V \in \mathfrak{X}(M)$ (vector field), $df(V) = d_V f$ (the directional derivative of f in direction of V)
- def: d is called <u>exterior derivative on forms</u>
  for $\omega \in \Omega^k(\mathbb{R}^n)$, $x^1, \dots, x^n$ coords let $I = (i_1, \dots, i_k)$ multi card
  $\omega = \sum \omega_I\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$, $\omega_I: \mathbb{R}^n \to \mathbb{R}$
  $d\omega_I \in \Omega^1(\mathbb{R}^n)$, define $d\omega = \sum (d\omega_I) \wedge dx^I$

$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0$. $d^2 = 0$, $\mathrm{im}\, d \subseteq \ker d$
- def: $H^k_{DR}(M) = \dfrac{\ker(d: \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(d: \Omega^{k-1} \to \Omega^k(M))} = \dfrac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$
  k^{th} deRham cohomology

- def: $d\omega = 0$ then ω is closed ("co-cycle") (kernel)
  $\omega = d\eta$ then ω is exact ("co-boundary") (image)
- note: 1) $H^k_{DR}(M) = 0$ if $k > \dim M$ or $k < 0$.
  2) $H^0_{DR}(M) = \ker(d : \underbrace{\Omega^0(M)}_{C^\infty(M)} \to \Omega^1(M))$ = locally constant fns (on each component)
  $= \mathbb{R}^{(\#\text{ components of } M)}$ ← $\mathbb{R}$ many constant fns, for each component
- Fact: $H^k_{DR}(M)$ is finite dimensional v.s. if M is compact
- thm: $f : M \to N$,

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{f^*} & \Omega^k(M) \\ \downarrow d & \circ & \downarrow d \\ \Omega^{k+1}(N) & \xrightarrow{f^*} & \Omega^{k+1}(M) \end{array}$$

commutes | if $[\omega] \in H^k_{DR}(N)$, $d\omega = 0$, $df^*\omega = f^*d\omega = 0$ so $[f^*\omega] \in H^k_{DR}(M)$

- lemma: $\omega \in \Omega^k(M)$, $v_1, \dots, v_{k+1}$ vector fields then
  $$d\omega(v_1, \dots, v_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} v_i(\omega(v_1, \dots, \hat{v}_i, \dots, v_{k+1})) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{k+1})$$
  ← Lie bracket
- ex: 1) $\omega \in \Omega^0(M) = C^\infty(M)$ $d\omega(v) = v \cdot \omega$
  2) $\omega \in \Omega^1(M)$ then $d\omega(v_1, v_2) = v_1 \cdot \omega(v_2) - v_2 \cdot \omega(v_1) - \omega([v_1, v_2])$

- generalize Lie derivative: $\omega \in \Omega^k(M)$, $v \in \mathfrak{X}(M)$, φ_t flow of v
  $\mathcal{L}_v \omega(x) := \lim_{t \to 0} (\varphi_t^* \omega_{\varphi_t(x)} - \omega_x)/t = \frac{d}{dt}(\varphi_t^* \omega_t)_{t=0}$
- lemma: $\mathcal{L}_v : \Omega^k(M) \to \Omega^k(M)$ "
  1) linear 2) $\mathcal{L}_v(\omega \wedge \eta) = (\mathcal{L}_v \omega) \wedge \eta + \omega \wedge (\mathcal{L}_v \eta)$ (product rule)
  3) $\mathcal{L}_v(\iota_w \omega) = \iota_{\mathcal{L}_v w} \omega + \iota_w \mathcal{L}_v \omega$
  4) $\mathcal{L}_v(\omega(v_1, \dots, v_k)) = (\mathcal{L}_v \omega)(v_1, \dots, v_k) + \sum_{i=1}^k \omega(v_1, \dots, \mathcal{L}_v v_i, \dots, v_k)$
- Cor: $\mathcal{L}_v(df) = d\mathcal{L}_v f$ (for functions f)
- ★ → Cartan's Magic Formula: for any k-form ω and vector field v:
  $$\mathcal{L}_v \omega = d\iota_v \omega + \iota_v d\omega$$
- Cor: $\mathcal{L}_v d = d\mathcal{L}_v$
- thm: v vector field on mfld, k-form ω invariant under flow of v ($\varphi_t^* \omega = \omega$) iff $\mathcal{L}_v \omega = 0$.

$\iota_v \omega$ is a $(k-1)$-form defined $(\iota_v \omega)(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1})$ "interior product" "contraction"

Integration

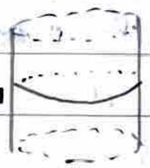
– Orientations –

- def: V vector space, 2 ordered bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ of V give the <u>same orientation</u> if the unique linear map $L: V \to V$, $v_i \mapsto w_i$ has + det.
  ↳ is an equivalence class on set of all ordered bases
  ↳ two equivalence classes (either + or −)
- def: a choice of equivalence class is an <u>orientation</u> on V
  ↳ for $V = \{0\}$, orientation is a choice of + or −
- orientation on V induces an orientation for V^*, $\Lambda^k V$
- If $\dim V = n$, $\Lambda^n V \cong \mathbb{R}$ (one dim vec space) so a choice of component of $\Lambda^n V - \{0\}$ is equiv. to a choice of orientation on $\Lambda^n V$ (so on V)
- def: let M be n-mfld, $\mathcal{O} = \Lambda^n M - \{0\} = \coprod_{p \in M} (\Lambda^n T_p M - \{0\})$
  $\mathcal{O}$ is a $(\mathbb{R} - \{0\})$-bundle over M. ← image of 0 in each fiber

each fiber of $\mathcal{O}$ has 2 comps, so if M connected, $\mathcal{O}$ has either 1 or 2 components.

$S_1 \times \mathbb{R}$

$S_1 \times \{0\}$ →



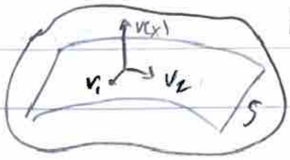
2 components

but for Möbius band

1 component

- If $\mathcal{O}$ has 2 components, then M is <u>orientable</u> and a choice of comp. is an orientation
  ↳ orientation on M gives orientation on each $T_x M$.
- If M orientable and connected, a choice of orientation on $T_x M$ (at a single pt) determines whole orientation on M.
- def: M, N oriented n-mflds, $f: M \to N$ is <u>orientation preserving</u> if it takes the component of $\mathcal{O}(N)$ determining orientation to the comp of $\mathcal{O}(M)$ determing orientation on M.
  ↳ $df_x: T_x M \to T_{f(x)} N$ takes oriented bases for $T_x M$ to oriented bases of $T_{f(x)} N$
- thm: M sm mfld. TFAE:
  a) M orientable
  b) $\exists$ collection charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$, $\{U_\alpha\}$ cover M, $\det(d(\varphi_\beta \circ \varphi_\alpha^{-1})_x) > 0$
  c) $\exists$ nowhere zero n-form on M

- ex: $f: M \to M$, $\omega \in \Omega^n(M)$ gives an orientation on M. Show f orientation preserving $\Leftrightarrow$ $f^*\omega$ is a pos. multiple of $\omega_{f(x)}$ $\forall x$
- lemma: M sm ortd mfld, $S \subseteq M$ submfld codim 1, V vector field along S transverse to S (V is a section of $TM|_S$ and T_xS, $V(x)$ span T_xM $\forall x \in S$)
  then: $\exists!$ orientation on S st $(v_1, \dots, v_{n-1})$ is an oriented basis for T_xS $\Leftrightarrow$ $(V(x), v_1, \dots, v_{n-1})$ is an oriented basis for T_xM $\forall x$

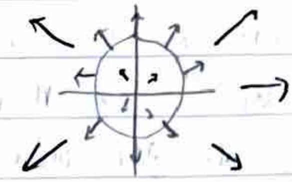


M (v_1, v_2) oriented bases for T_xS

$\Leftrightarrow$

$(V(x), v_1, v_2)$ ortd bases for T_xM

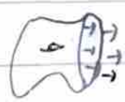
- ex: $S^n \subset \mathbb{R}^{n+1}$



  $V = \sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}$, V is transverse to S^n
  $(S^n = f^{-1}(1)$, $f(x_1, \dots, x_n) = \sum (x_i)^2)$
  $\Omega = dx^1 \wedge \dots \wedge dx^{n+1}$
  nonzero $(n+1)$ form on $\mathbb{R}^{n+1}$. lemma: $\omega = i_V \Omega = \sum (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$ ↗ gives orientation on S^n
- ex: $r: S^n \to S^n$; $p \mapsto -p$ orientation preserving iff n odd
  $(r^*\omega = (-1)^{n+1}\omega)$

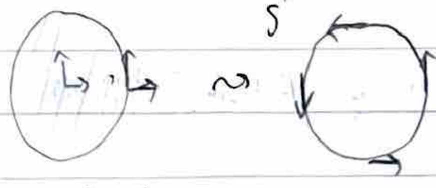
★ n-mfld orientable means $\exists$ n-form $(\omega \in \Omega^n(M))$ that never vanishes ★

- can construct vector field on M which points outwards on ∂M



  Ω an n-form which orients M, then $i_V M$ orients ∂M
  $(v_1, \dots, v_{n-1})$ orients $T_x\partial M$ $\Leftrightarrow$ $(V(x), v_1, \dots, v_{n-1})$ orients T_xM

D^2 S^1



– Integration –

- $x^1, \dots, x^n$ coords on $\mathbb{R}^n$, $\Lambda^n \mathbb{R}^n \cong \mathbb{R}$ so n-forms are $\omega = f(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n$ (for some $f: \mathbb{R}^n \to \mathbb{R}$)
- def: $A \subseteq \mathbb{R}^n$, define $\int_A \omega = \int_A f \, d\mathrm{vol}$ ← from vec calculus
  can we extend to mflds? have to see what happens w/ coord charts/change
  ↳ yes, works out ✓

- def: $\omega \in \Omega^n(M^n)$, M oriented. if ω supported in U open, $\exists$ coord chart $\varphi: U \to V$ then define $\int_M \omega = \int_{\mathbb{R}^n} (\varphi^{-1})^* \omega$
  - ↳ well defined if φ orientation preserving
- def: $\omega \in \Omega^n(M^n)$, M oriented, so can take collection of coord charts orn preserving $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$, $M = \bigcup_\alpha U_\alpha$, $\{\psi_\alpha\}$ partition of unity
  $\int_M \omega = \sum_\alpha \int_M \psi_\alpha \omega = \sum_\alpha \int_{\mathbb{R}^n} (\varphi_\alpha^{-1})^* \psi_\alpha \omega$ (by prev def)
  ↳ need to show choice of cover / partition of unity doesn't change $\int$.
- Lemma: integration is independent of cover / partition of unity.
  $\int_A \omega = \sum_\alpha \int_{\varphi_\alpha(A \cap U_\alpha)} (\varphi_\alpha^{-1})^* \psi_\alpha \omega$ for $A \subseteq M$
- thm: 1) Integral is linear
  2) swapping orn introduces negative
  3) for orn pres. diffeo $f: N \to M$, $\int_M f^* \omega = \int_N \omega$
  4) $M = M_1 \cup M_2$, $M_1 \cap M_2$ (dim M)-1 mfld in M then $\int_M \omega = \int_{M_1} \omega + \int_{M_2} \omega$
  5) $\int_M \omega = \int_{M-A} \omega$ if A ms 0.
- how to integrate fns. on orientable mfld M?
  Choose a non zero n form ω on M (defines an orientation)
  def: ω is called a volume form on M, denoted $d$vol (this is not d of something)
  def: for $f: M \to \mathbb{R}$, $\int_M f = \int_M f \, d\text{vol}$
  def: the volume of M is $\int_M d\text{vol}$ (dependent on $d$vol)
- def, $\alpha \in \Omega^k(M^n)$, $\Sigma \subseteq M$ k-dim submfld, $i: \Sigma \hookrightarrow M$ inclusion. def: $\int_\Sigma \alpha = \int_\Sigma i^* \alpha$

— Stokes Thm —

- thm: M sm oriented n-mfd w/ boundary. let $\beta \in \Omega^{n-1}(M)$ that is compactly supported. then $\boxed{\int_M d\beta = \int_{\partial M} \beta}$
  ↳ FTC: $\int_{[a,b]} \underbrace{f'(t)\,dt}_{= df(t)} = \int_{\{a,b\}} f = f(b) - f(a)$
- Cor: M sm compact n-mfld
  1) If $\partial M = \emptyset$, $\int_M d\beta = 0$ $\forall \beta \in \Omega^{n-1}(M)$
  2) $\omega \in \Omega^{n-1}(M)$ closed, $\int_{\partial M} \omega = 0$
  3) ω closed k-form, S oriented k-submfld, $\partial S = \emptyset$, then ($\int_S \omega \neq 0 \Rightarrow$ ω not exact, & S does not bound a $(k+1)$ dim submfld of M

- Cor: $f_0, f_1 : \Sigma \to M$ sm, Σ oriented k-mfld, f_0 homotopic to f_1 rel $\partial\Sigma$ and α closed k-form on M, then $\int_\Sigma f_0^*\alpha = \int_\Sigma f_1^*\alpha$

— End of Final Exam Content —

– De Rham Cohomology –

$$0 \xrightarrow{d_0} \Omega^0(M) \xrightarrow{d_1} \Omega^1(M) \to \dots \xrightarrow{d_n} \Omega^n(M) \qquad d^2 = 0$$

$H^k_{DR}(M) = \ker(d)/\mathrm{im}(d)$ = closed k-forms / exact k-forms

$f: M \to N$ induces $f^*: H^k_{DR}(N) \to H^k_{DR}(M)$

$(f\circ g)^* = g^* \circ f^*$, M diffeo $N \Rightarrow H^k_{DR}(M) \cong H^k_{DR}(N)$

$H^0_{DR}(M) = \mathbb{R}^{\#\text{components of } M}$

- Thm: homotopic maps induce same map on cohomology
- def: $f: M \to N$ is homotopy equivalence if $\exists g: N \to M$ st. $g\circ f$, $f\circ g$ homotopic to id.
- Thm (Mayer-Vietoris): M sm mfld, U, V open st $M = U \cup V$. for each k, $\exists$ linear map $\delta^k : H^k_{DR}(U\cap V) \to H^{k+1}_{DR}(M)$ st.

$$\dots \to H^k_{DR}(M) \xrightarrow{k^* \oplus \ell^*} H^k(U) \oplus H^k(V) \xrightarrow{i^* - j^*} H^k(U\cap V) \xrightarrow{\delta^k} H^{k+1}(M) \to \dots$$ is exact and

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U\cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "\ell"'] & \end{tikzcd}$$ are inclusion maps.

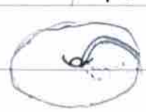
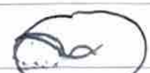
- $H^k_{DR}(S^n) = \{ \mathbb{R}\ \ k = 0, n \ ;\ 0 \ \ k \neq 0, n \qquad n \geq 1$

and $H^n(S^n)$ is generated by any volume form of S^n

- thm: M compact, connected, $\partial M = \emptyset$, oriented n-mfld,

$I: H^n_{DR}(M) \to \mathbb{R}$ by $[\omega] \mapsto \int_M \omega$ is an isomorphism

- $H^k_{DR}(T^2) = \{ \mathbb{R}\ k=2,\ \mathbb{R}\oplus\mathbb{R}\ k=1,\ \mathbb{R}\ k=0,\ 0$ else

$T_2 = S^1 \times S^1 \quad U =$  $\quad V =$  $\quad U \simeq V \simeq S^1 \times \mathbb{R}$

$\Rightarrow H^k(U) \cong H^k(V) \cong H^k(S^1) = \{ \mathbb{R}\ k = 0,1 \quad 0$ else

$H^k(U\cap V) = H^k(\bigsqcup_2 (S^1 \times \mathbb{R})) \cong \{ \mathbb{R}\oplus\mathbb{R}\ k = 0,1 \quad 0$ else

$\mathbb{R} \cong A/\mathbb{R} \Rightarrow A = \mathbb{R}\oplus\mathbb{R}$ for vector spaces, not true for groups in general (e.g. $\mathbb{Z}$)

— Products —

- $H^*_{DR}(M) = \bigoplus_{i=0}^{n} H^i_{DR}(M)$ is a ring w/ product $[\omega] \cup [\eta] = [\omega \wedge \eta]$
grading $[\alpha] \cup [\beta] = (-1)^{|\alpha||\beta|} [\beta] \cup [\alpha]$.
$f: M \to N$ induces ring homo $f^*: H^*_{DR}(N) \to H^*_{DR}(M)$
- Poincaré Duality : M compact oriented connected n-mfld, $\partial M = \emptyset$.
$H^k_{DR}(M) \cong H^{n-k}_{DR}(M)$.