- $P[a,b]$ is the set of all partitions $\Gamma = \{x_0, \dots, x_n\}$ of $[a,b]$
- $S_\Gamma[f;a,b] = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$
- $V[f;a,b] = \sup\{S_\Gamma[f;a,b] : \Gamma \in P[a,b]\}$
- f is of <u>bounded variation</u> on $[a,b]$ if $V[f;a,b] < \infty$
- $x^+ = \max\{x, 0\}$; $x^- = -\min\{0, x\}$
- $P_\Gamma[f;a,b] = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+$ $\quad N_\Gamma[f;a,b] = \sum_{i=1}^{n} (\dots)^-$
- <u>positive variation</u> of f over $[a,b]$ is $P[f;a,b] = \sup\{P_\Gamma[f;a,b] : \Gamma \in P[a,b]\}$

  <u>negative variation</u> " " $N[f;a,b] =$ " N_Γ "

- $*$ $x^+, x^- \geq 0$, $|x| = x^+ + x^-$ $\quad x = x^+ - x^-$
- $S_\Gamma = P_\Gamma + N_\Gamma$ $\quad P_\Gamma - N_\Gamma = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a)$ (Telescoping)
- <u>thm</u>: let $f: [a,b] \to \mathbb{R}$
  a) if any of $P[f;a,b]$, $N[f;a,b]$, $V[f;a,b]$ are finite, then all are
  b) $P[f;a,b] + N[f;a,b] = V[f;a,b]$
  c) $P[f;a,b] - N[f;a,b] = f(b) - f(a)$

(1)

Proof: a) Note $P_\Gamma[f;a,b] + N_\Gamma[f;a,b] = S_\Gamma[f;a,b]$ and

(2) $P_\Gamma[f;a,b] - N_\Gamma[f;a,b] = f(b) - f(a)$. Since $P_\Gamma, N_\Gamma \geq 0$,
we know $0 \le P_\Gamma \le S_\Gamma \le V_\Gamma$ and $0 \le N_\Gamma \le S_\Gamma \le V_\Gamma$.
Taking sup over $\Gamma \in \mathcal{P}[a,b]$ gives $0 \le P[f;a,b] \le V[f;a,b]$
and $0 \le N[f;a,b] \le V[f;a,b]$, so if $V[f;a,b] < \infty$, the other two are as well.

$(1)+(2) \Rightarrow 2P_\Gamma[f;a,b] = S_\Gamma[f;a,b] + (f(b)-f(a))$.
Taking sup over all $\Gamma \in \mathcal{P}[a,b]$, $2P[f;a,b] = V[f;a,b] + (f(b)-f(a))$ (3)
So if $P[f;a,b] < \infty$, $V[f;a,b]$ is as well, hence $N[f;a,b] < \infty$.
$(1)-(2) \overset{\sup}{\Rightarrow} 2N[f;a,b] = V[f;a,b] - (f(b)-f(a))$ (4) if $N<\infty$, $P<\infty$ and $V<$

b) (3)+(4). c) $(3)-(4) \Rightarrow 2P[f;a,b] - 2N[f;a,b] = 2(f(b)-f(a))$ □

- If $[a',b'] \subseteq [a,b]$ then $V[f;a',b'] \le V[f;a,b]$
- Likewise, $P[f;a',b'] \le P[f;a,b]$ and $N[f;a',b'] \le N[f;a,b]$
- Jordan's Thm: Let $f:[a,b] \to \mathbb{R}$. f is of bounded variation on $[a,b]$ iff $f = g-h$, g,h are bounded increasing fns on $[a,b]$

Pf: $(\Leftarrow)$: Let $f = g-h$, g,h bounded increasing. So g,h are bounded variation. The diff of bnd. variation are bound variation, so f is of bound variation.

$(\Rightarrow)$: Let $f \in BV[a,b]$. Let $P(x) = P[f;a,x]$ and $N(x) = N[f;a,x]$.

$x \in [a,b]$ If $a \le x < y \le b$, then $[a,x] \subseteq [a,y]$, so $0 \le P(x) = P[f;a,x] \le P[f;a,y] = P(y)$.
Similarly, $N(x) \le N(y)$. So P, N are increasing. Also,
$0 \le P(x) = P[f;a,x] \le P[f;a,b] < \infty$ (by prev thm)
$0 \le N(x) = N[f;a,x] \le N[f;a,b] < \infty$

Define $g(x) = P(x) + f(a)$, $h(x) = N(x)$. Then

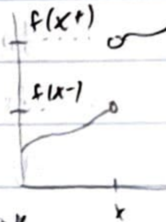
$$g - h = (P(x) + f(a)) - N(x) = P[f;a,x] - N[f;a,x] + f(a)$$
$$= \underbrace{f(x) - f(a)}_{\text{by prev thm}} + f(a) = f(x)$$ □

- thm 2.8: Let $f: [a,b] \to \mathbb{R}$ have bounded variation on $[a,b]$. Then f has countably many discontinuities and they are jump discontinuities.

Proof: As $f = g - h$ for g, h bounded increasing, it suffices to show this only for bounded monotone increasing fns (if we show this, then g, h have disconts countable, so difference is still countable disconts)

So assume f is monotone increasing, bounded. Then for $x \in (a,b)$,
$f(x-) = \lim_{t \uparrow x} f(t)$; $f(x+) := \lim_{t \downarrow x+} f(t)$ exist and $f(x-) \leq f(x^+)$ (monotonicity).

Hence f is discont at x only when $f(x^-) < f(x+)$ (jump discont only).



Suppose x is jump discont. $\exists\, r_x \in \mathbb{Q} \cap (f(x-), f(x+))$.
As f is increasing, distinct jump disconts leads to distinct r_x. Then $x \mapsto r_x$ is one-to-one, but rationals are countable.
So the set of disconts is countable. □

- Notation: for partition $\Gamma = \{x_0, \dots, x_m\}$, $|\Gamma| = \max\{x_j - x_{j-1} : 1 \leq j \leq m\}$
- Cor 2.10: If f' cont on $[a,b]$, then $V[f; a,b] = \int_a^b |f'(t)|\,dt$

Pf: $(\leq)$: Let $\Gamma = \{x_0, \dots, x_m\} \in P[a,b]$. Then $S_\Gamma[f; a,b] = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$
$= \sum^{m} \left|\int_{x_{i-1}}^{x_i} f'(t)\,dt\right|$ (by FTC) $\leq \sum^{m} \int_{x_{i-1}}^{x_i} |f'(t)|\,dt = \int_a^b |f'(t)|\,dt$.
Take sup over $\Gamma \in P[a,b]$, $V[f; a,b] \leq \int_a^b |f'(t)|\,dt$

$(\geq)$: Let $\varepsilon > 0$. Since f' is cont, we can find $\Gamma = \{x_0, \dots, x_m\} \in P[a,b]$ s.t. $|\Gamma|$ small enough so $\sum_{i=1}^{m} |f'(\xi_i)|(x_i - x_{i-1}) \geq \int_a^b |f'(t)|\,dt - \varepsilon$ for $\xi_i \in [x_{i-1}, x_i]$ $\forall i$. Now choose ξ_i s.t. $f(x_i) - f(x_{i-1}) = f'(\xi_i)(x_i - x_{i-1})$ (MVT)
Then $V[f; a,b] \geq S_\Gamma[f; a,b] = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| = \sum_{i}^{m} |f'(\xi_i)|(x_i - x_{i-1})$
$\geq \int_a^b |f'(t)|\,dt - \varepsilon$. Since ε was arb, $V[f; a,b] \geq \int_a^b |f'(t)|\,dt$. □

- Riemann-Stieltjes Integral:

def: Let $f, \phi: [a,b] \to \mathbb{R}$

a) Let $\Gamma = \{x_0, \dots, x_m\} \in P[a,b]$. Define R-S sum as
$R_\Gamma = \sum_{i=1}^{m} f(\xi_i)(\phi(x_i) - \phi(x_{i-1}))$ where $\xi_i \in [x_{i-1}, x_i]$

b) If $I = \lim_{|\Gamma| \to 0} R_\Gamma$ exists, then f is R-S integrable w.r.t. ϕ and write $I = \int_a^b f(x)\,d\phi(x)$ or $\int_a^b f\,d\phi$.

- Remarks:
  a) If $\phi(x) = x$, then you get Riemann Integral
  b) If $\phi'(x)$ is cont, $I = \int_a^b f(x)\,\phi'(x)\,dx$.
  c) If f cont, $\phi \in BV[a,b]$, then $\int_a^b f(x)\,d\phi(x)$ exists
  d) IBP: $\int_a^b f\,d\phi$ exists iff $\int_a^b \phi\,df$ exists;
     then $\int_a^b f\,d\phi = \left((f\phi)(b) - (f\phi)(a)\right) - \int_a^b \phi\,df$
- Interval (box): $I = \{\vec{x} = (x_1, \dots, x_n) : a_j \leq x_j \leq b_j \;\; \forall j\}$
  $vol(I) = \prod_{j=1}^{n} (b_j - a_j)$
- Let $S = \{I_k : k \geq 1\}$ be a countable collection of intervals
  Define $\sigma(S) = \sum_{I \in S} vol(I)$.
- Let $E \subset \mathbb{R}^n$. S covers E if $E \subset \bigcup_k I_k$.
- Lebesgue outer measure or exterior measure of E is
  $|E|_e = \inf\{\sigma(S) : S \text{ covers } E\}$
- boundary of any interval in $\mathbb{R}^n$ has exterior measure 0.
- If $|E|_e < \infty$ and $\varepsilon > 0$, then $\exists\, S = \{I_k\}_{k \in \mathbb{N}}$ that covers E and $|E|_e \leq \sigma(S) < |E|_e + \varepsilon$
- Let $E \subseteq \mathbb{R}^n$ be countable. Then $|E|_e = 0$.
  Pf: Cover each pt w/ a small enough box $\left(\frac{\varepsilon}{2^j}\right)$ □
- the Lebesgue outer measure is translation invariant.

Lemma A
- Let $I, \{I_k\}_{k=1}^{N}$ be intervals w/ $I \subset \bigcup I_k$. Then $v(I) \leq \sum_k^N v(I_k)$

Pf: Improper Riemann Integral. Let
$$\chi_I(\vec{x}) = \begin{cases} 0 & \vec{x} \notin I \\ 1 & \vec{x} \in I \end{cases}$$
. Then $\chi_I(\vec{x}) \leq \sum^N \chi_{I_k}(\vec{x})$.
$$vol(I) = \int_{\mathbb{R}^n} \chi_I(\vec{x})\,d\vec{x} \leq \int_{\mathbb{R}^n} \sum^N \chi_{I_k}(\vec{x})\,d\vec{x} = \sum^N \int_{\mathbb{R}^n} \chi_{I_k}(\vec{x})\,d\vec{x}$$
$$= \sum^N vol(I_k)$$ □

- Thm 3.2: $|I|_e = vol(I)$ (I interval in $\mathbb{R}^n$)
  Pf: ($\leq$): Since I covers itself, $|I|_e \leq \sigma(\{I\}) = |vol(I)|$.
  ($\geq$): Let $S = \{I_k\}_{k \in \mathbb{N}}$ cover I. Let $\varepsilon > 0$. For $k \geq 1$, let I_k^* be an interval containing I_k in its interior st. $vol(I_k^*) \leq (1+\varepsilon)\,vol(I_k)$

Then $I_k^* \subseteq \bigcup_{k=1}^{\infty} I_k \subseteq \bigcup_{k=1}^{\infty} \overbrace{\mathrm{int}(I_k^*)}^{(I_k^*)^\circ}$. I is closed and bounded, hence compact, so there is a finite subcover, $I \subseteq \bigcup_{k=1}^{N} \mathrm{int}(I_k^*)$. By Lemma A, $\mathrm{vol}(I) \leq \sum^{N} \mathrm{vol}(I_k^*) \leq (1+\varepsilon) \sum_k^N \mathrm{vol}(I_k) \leq (1+\varepsilon) \sum_k^\infty \mathrm{vol}(I_k) = (1+\varepsilon)\sigma(S)$

Taking inf over all S covering I, $\mathrm{vol}(I) \leq (1+\varepsilon)|I|_e$ but $\varepsilon > 0$ was arbitrary so $\mathrm{vol}(I) \leq |I|_e$. $\square$

- Thm 3.3: $A \subset B \Rightarrow |A|_e \leq |B|_e$. (monotone)

Pf: If $\{I_k\}_k^{=S}$ covers B, then it covers A. So inf in A is bounded above by $\sigma(S)$, take inf. $\square$

- Thm 3.4: $|\cdot|_e$ is countably subadditive

Pf: $E = \bigcup_k^\infty E_k$. If $|E_k|_e = \infty$ for any, then is trivial. So assume all $|E_k|_e$ are finite. Let $\varepsilon > 0$. We can find countably # intervals so: for each k, $E_k \subset \bigcup_j^\infty I_j^{(k)}$ and $\sum^\infty \mathrm{vol}(I_j^{(k)}) \leq |E_k|_e + \frac{\varepsilon}{2^k}$.

Taking all of these intervals, $S = \{I_j^{(k)} : \forall j, k\}$ is countable that covers E.

So $|E|_e \leq \sigma(S) = \sum_k \sum_j \mathrm{vol}(I_j^{(k)}) \leq \sum_k (|E_k|_e + \frac{\varepsilon}{2^k}) = \left(\sum_k |E_k|_e\right) + \varepsilon$

and $\varepsilon > 0$ was arb. $|E|_e \leq \sum_k |E_k|_e$ $\square$

- Cantor set: $C = \bigcap C_n$, C_n is C_{n-1} w/ middle thirds removed.

$C_n = \bigcup^{2^n} I_{n,k}$ and $C \subseteq C_n$ $\forall n$ (ternary expansion → binary → reals)

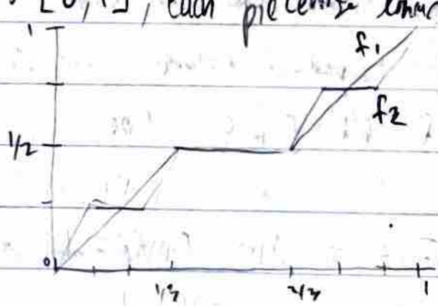
$|C|_e \leq \sum_{k=1}^{2^n} \mathrm{vol}(I_{n,k}) = 2^n/3^n \xrightarrow{(n\to\infty)} 0$ so $|C|_e = 0$. Cantor set is uncountable

also: compact, perfect (every pt in C is a limit pt), $\mathrm{int}(C) = \emptyset$

- Cantor Lebesgue Function: $f_n : [0,1] \to [0,1]$, each piecewise linear & monotone inc. and f_n constant outside of C_n

$f = \lim f_n$

- f is continuous, monotone increasing
- $f(0) = 0$, $f(1) = 1$
- f' exists in each open interval in $[0,1] \setminus C$ and $= 0$ there. So $f' = 0$ except on a set of $\underset{\text{measure } 0}{\text{Leb. out}}$.



- thm 3.6: outer regularity of Lebesgue measure: let $E \subseteq \mathbb{R}^n$, $\varepsilon > 0$. $\exists\, G \subseteq \mathbb{R}^n$ open, $E \subseteq G$ st. $|G|_e \leq |E|_e + \varepsilon$.

Consequently, $|E|_e = \inf\{|G|_e : G \text{ open}, E \subseteq G\}$

Pf: If $|E|_e = \infty$, take $G = \mathbb{R}^n$.
So assume $|E|_e < \infty$ and $\varepsilon > 0$. Choose intervals $\{I_k\}_{k \in \mathbb{N}}$ s.t. $E \subset \bigcup_{k\in\mathbb{N}} I_k$ and $\sum_{k=1} vol(I_k) \le |E|_e + \frac{\varepsilon}{2}$. Let I_k^* be an interval containing I_k in its interior (slightly larger), with $vol(I_k^*) < vol(I_k) + \frac{\varepsilon}{2^{k+1}}$. Let $G = \bigcup_{k\in\mathbb{N}} int(I_k^*)$, then $E \subseteq \bigcup_{k\in\mathbb{N}} I_k \subseteq G$.
Also $G = \bigcup_{k\in\mathbb{N}} int(I_k^*) \subseteq \bigcup_{k\in\mathbb{N}} I_k^*$. Then
$|G|_e \le \sum_{k\in\mathbb{N}} vol(I_k^*) < \sum_{k\in\mathbb{N}} \left(vol(I_k) + \frac{\varepsilon}{2^{k+1}}\right) \le \left(|E|_e + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = |E|_e + \varepsilon$. □

- def: G_δ set: countable intersection of open sets $(\mathbb{R} - \mathbb{Q})$
  F_σ set: countable union of closed sets $(\mathbb{Q})$
- thm 3.8: Let $E \subseteq \mathbb{R}^n$. $\exists$ a G_δ set H st. $E \subset H$ and $|E|_e = |H|_e$.
Pf: If $|E|_e = \infty$, take $H = \mathbb{R}^n$.
So assume $|E|_e < \infty$. By thm 3.6, $\exists$ open $G_k \supseteq E$ st. $|G_k|_e < |E|_e + \frac{1}{k}$
let $H = \bigcap_{k\in\mathbb{N}} G_k$. Then H is a G_δ set and $E \subseteq H$ (as each $E \subseteq G_k$).
Also, $E \subseteq H \subseteq G_k$, so $|E|_e \le |H|_e \le |G_k|_e < |E|_e + \frac{1}{k}$. Since H is independent of k, $|E|_e \le |H|_e \le |E|_e \Rightarrow |E|_e = |H|_e$. □
- def: Let $E \subseteq \mathbb{R}^n$. E is Lebesgue measurable if for all $\varepsilon > 0$, there is an open set $G \supset E$ st. $|G \setminus E|_e < \varepsilon$. If E is λ-measurable, then its measure $|E| = |E|_e$.
- Opensets are measurable: Pf. let E be open, take $G = E$. □
- If $|E|_e = 0$, then E is m-ble: Pf: let $\varepsilon > 0$, By thm 3.6, $\exists$ open $G \supset E$ st. $|G|_e < |E|_e + \varepsilon = \varepsilon$. Then $G \setminus E \subset G$, so $|G \setminus E|_e \le |G| < \varepsilon$. □
- thm 3.12: Countable union of m'ble sets are m'ble.
Pf: let $E_k \subseteq \mathbb{R}^n$ be m'ble $\forall k \ge 1$. Let $E = \bigcup_{k\in\mathbb{N}} E_k$. Choose open $G_k \supseteq E_k$ st. $|G_k \setminus E_k|_e < \varepsilon/2^k$ (since E^k m'ble). Let $G = \bigcup_{k\in\mathbb{N}} G_k$, then $E \subseteq G$. Also, $G \setminus E = \left(\bigcup_{k\in\mathbb{N}} G_k\right) - \left(\bigcup_{j\in\mathbb{N}} E_j\right) = \bigcup_{k=1}^{\infty}\left(G_k - \bigcup_j E_j\right) \subseteq \bigcup_{k\in\mathbb{N}} (G_k \setminus E_k)$
so by thm 3.4, $|G \setminus E|_e \le \sum_{k\in\mathbb{N}} |G_k \setminus E_k| < \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^k} = \varepsilon$, so E is m'ble. □
by thm 3.4, $|E| = |E|_e \le \sum_{k=1} |E_k|_e = \sum_{k\in\mathbb{N}} |E_k|$. □
- Corollary 3.13: Any interval I is m'ble and $|I| = vol(I)$
Pf: Note $I = int(I) \cup \partial I$. $int I$ is open and $|\partial I|_e = 0$ so ∂I is m'ble.
By thm 3.12, I is m'ble, then $|I| = |I|_e = vol(I)$. □

thm 3.14: every closed set is m'ble

Lemma 3.15: Let $\{I_k\}_{k=1}^{N}$ be nonoverlapping intervals (only overlap on boundary). Let $E = \bigcup_k^N I_k$, then E is m'ble and $|E| = \sum_k^N |I_k|$

$\frac{1}{2}$ PF: ($\le$): Each I_k m'ble, so E m'ble, by thm 3.12, $|E| \le \sum_k^N |I_k|$

($\ge$): Cover E by intervals, intersect them with I_k's. □

Lemma 3.16: Let $A, B \subseteq \mathbb{R}^n$ with distance between A, B being positive. Then $|A \cup B|_e = |A|_e + |B|_e$.

$\frac{1}{2}$ PF: Let d = dist. from A to B. Cover A, B w/ intervals w/ diam $< d$. Then no interval can intersect both A and B. Divide intervals into those $S_1 \to$ intervals covering A, $S_2 \to$ intervals covering B or nothing. □

Lemma 3.16a: Let $F_1, \dots, F_N$ be disjoint, compact. Then $|\bigcup_{j=1}^N F_j|_e = \sum_{j=1}^N |F_j|_e$

Pf: Since $\{F_j\}_{j=1}^N$ is compact and disjoint, the distance between any two is positive. Apply lemma 3.16 repeatedly. □

Proof: Suppose first that F is bounded and closed, hence compact. Let $\varepsilon > 0$, by thm 3.6, $\exists G \supseteq F$ open st. $|G|_e < |F|_e + \varepsilon$. Then $G \setminus F = G \cap F^c$ is open. So $G \setminus F = \bigcup_{k \in \mathbb{N}} I_k$ (countable union of intervals) and I_k's nonoverlapping.
By thm 3.4, $|G \setminus F|_e \le \sum_{k \in \mathbb{N}} |I_k|_e$. Then for all $N \ge 1$, $G = F \cup \left(\bigcup_{k=1} I_k\right) \supseteq F \cup \left(\bigcup_{k=1}^N I_k\right)$ then F, $\bigcup_{k=1}^N I_k$ are disjoint compact sets. Then $|G|_e \ge |F \cup (\bigcup_{k=1}^N I_k)|$ (monotonicity).
$= |F|_e + |\bigcup_{k=1}^N I_k| = |F|_e + \sum_{k=1}^N |I_k|_e \Rightarrow \sum_{k=1}^N |I_k|_e \le |G|_e - |F|_e < \varepsilon$.
(disjoint compact sets, L.3.16) (3.15)
But N was arbitrary, so $\sum_{k \in \mathbb{N}} |I_k| < \varepsilon$.
So $|G \setminus F|_e = |\bigcup_{k \in \mathbb{N}}^{\infty} I_k|_e \le \sum_{k \in \mathbb{N}} |I_k|_e < \varepsilon$. closed, bounded $\Rightarrow$ compact & m'ble
If F is unbounded, we can write $F = \bigcup_{n=1}^{\infty} (F \cap cl(B_n(0)))$, so by thm 3.12, F is m'ble. □

thm 3.17: Complements of m'ble sets are m'ble

Proof: for each $k \ge 1$, choose open $G_k \supseteq E$ that $|G_k \setminus E|_e < \frac{1}{k}$.
As G_k is open, G_k^c is closed and hence measurable. Let $H = \bigcup_{k \in \mathbb{N}} G_k^c$, which is m'ble. Next, $G_k \supset E$ so $E^c \supseteq G_k^c \Rightarrow H \subseteq E^c$.
Then let $Z = E^c \setminus H$. Then $E^c = Z \cup H$ (so we WTS $|Z|_e = 0$, hence m'ble).
$Z = E^c \setminus H = E^c \setminus \left(\bigcup_{k \in \mathbb{N}} G_k^c\right) \subseteq E^c \setminus G_k^c$ (for each $k \in \mathbb{N}$)
$= G_k \setminus E \Rightarrow |Z|_e \le |G_k \setminus E| < \frac{1}{k}$ $\forall k$, so $|Z|_e = 0$ as desired □

G_k E

- thm 3.18: Countable intersection of m'ble sets is m'ble.

  Proof: let $E = \bigcap_{k\in\mathbb{N}} E_k$ and E_k m'ble. So E_k^c is m'ble.

  Then $E^c = \bigcup_{k\in\mathbb{N}} E_k^c$ is also m'ble. Hence E is m'ble. $\square$

- thm 3.19: let A, B be m'ble. Then $A \setminus B$ is m'ble.

  Proof: $A \setminus B = A \cap B^c$ is m'ble. $\square$

- def: let Σ be a nonempty collections of subsets of $\mathbb{R}^n$. We say Σ is a σ-algebra if

  a) $E \in \Sigma \Rightarrow E^c \in \Sigma$ b) closed under countable union

- Remarks: then Σ is also
  - closed under countable intersections
  - $\emptyset \in E$. Pf: let $E \in \Sigma$ (nonempty), then $E^c \in \Sigma$ so $E \cap E^c = \emptyset \in \Sigma$

- thm 3.20: the collection $\mathcal{M}$ of all Lebesgue m'ble sets is a σ-algebra.

- Remarks:
  - G_δ and F_σ sets are m'ble
  - If E_k m'ble, then $\limsup_{k\to\infty} = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k$ (belong to inf. many E_k) m'ble

    $\liminf_{k\to\infty} = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k$ (belong to all but finitely many)
  - Intersections of σ-algebra is a σ-algebra

- def: the Borel sets $\mathcal{B}$ is the σ-alg generated by open sets.

  $\mathcal{B} = \bigcap \{\Sigma : \Sigma \text{ is a } \sigma\text{-alg of } \mathbb{R}^n \text{ containing all open sets}\}$

  → $\mathcal{B}$ is the smallest σ-alg containing the open sets

  → G_δ and F_σ are Borel sets

- thm 3.21: $\mathcal{B} \subseteq \mathcal{M}$

  Pf: $\mathcal{M}$ contains open sets so $\mathcal{B} \subseteq \mathcal{M}$. $\square$

- Lemma 3.22: the Lebesgue measure is inner regular: let $E \subseteq \mathbb{R}^n$. Then E is m'ble iff $\forall \varepsilon > 0$, $\exists$ closed $F \subseteq E$ s.t. $|E \setminus F|_e < \varepsilon$

  Proof: E m'ble, so E^c m'ble. So $\forall \varepsilon > 0$, $\exists$ open $G \supseteq E^c$ s.t. $|G \setminus E^c|_e < \varepsilon$ (Leb-measure is outer regular). Then G^c is closed, and $G^c \subseteq E$, and $|G \setminus E^c| = |E \setminus G^c| < \varepsilon$ as desired. $\square$

- If E m'ble: $|E| = \inf\{|G|_e : G$ open, $G \supseteq E\}$ (outer reg)
$= \sup\{|F|_e : F$ closed, $F \subseteq E\}$ (inner reg)

(Thm 3.23) • Countable additivity of Leb-measure: Let $\{E_k\}_{k\in\mathbb{N}}$ be disjoint, m'ble sets. Then $|\bigcup_{k=1}^{\infty} E_k| = \sum_{k\in\mathbb{N}} E_k$.

Proof: [we will choose closed F_k so that $|E_k \setminus F_k|$ is small]

Case 1: Assume E_k is bounded for $k \geq 1$. Let $\varepsilon > 0$. Then $\exists$ closed F_k s.t. $|E_k \setminus F_k| < \frac{\varepsilon}{2^k}$. So $E_k = F_k \cup (E_k \setminus F_k) \Rightarrow |E_k| \leq |F_k| + |E_k \setminus F_k| \leq |F_k| + \varepsilon/2^k \Rightarrow |F_k| \geq |E_k| - \varepsilon/2^k$. The E_k are bounded and disjoint so F_k is as well, hence compact. The dist between F_j and $F_k > 0$ if $j \neq k$. By Lemma 3.16A, $\sum_{k=1}^{N} |F_k| = |\bigcup_{k=1}^{N} F_k| \leq |\bigcup_{k=1}^{\infty} F_k| \leq |\bigcup_{k=1}^{\infty} E_k|$

Then as $N \to \infty$, $|\bigcup_{k=1}^{\infty} E_k| \geq |\bigcup_{k\in\mathbb{N}} F_k| \geq \sum_{k=1}^{\infty} (|E_k| - \frac{\varepsilon}{2^k}) = \sum_{k=1}^{\infty} |E_k| - \varepsilon$ (monotonicity)

since $\varepsilon > 0$ arb, we have $|\bigcup_{k=1}^{\infty} E_k| \geq \sum_{k=1}^{\infty} |E_k|$. By σ-subadditivity, the other $(\leq)$ is clear.

Case 2: Some $\{E_k\}$ are unbounded. For $j, k \geq 1$, let $E_k^j = (E_k \cap B_j(0)) \setminus (B_{j-1}(0))$. Then E_k^j m'ble, bounded, & are disjoint.

By case 1, $|\bigcup_{k\in\mathbb{N}} E_k| = |\bigcup_k \bigcup_j E_k^j| = \sum_k \sum_j |E_k^j| = \sum_k |E_k|$ by case 1 again. $\square$

- Cor 3.24: Let $\{I_k\}$ be a seq of non-overlapping intervals. Then $|\bigcup I_k| = \sum |I_k|$

Pf: By thm 3.12, $|\bigcup I_k| \leq \sum_k |I_k|$

$(\geq)$: Let $I_k = \operatorname{int}(I_k) \cup \partial I_k$ and $|\partial I_k|_e = 0$, so m'ble. By thm 3.23, $|I_k| = |\operatorname{int}(I_k)| + |\partial I_k| = |\operatorname{int}(I_k)|$. Then $\{\operatorname{int}(I_k)\}$ are disjoint, m'ble, so $|\bigcup I_k| \geq |\bigcup \operatorname{int}(I_k)| = \sum_k |\operatorname{int}(I_k)| = \sum_k |I_k|$ as desired. $\square$

- Cor 3.25: Let E_1, E_2 m'ble and $E_2 \subseteq E_1$ and $|E_1| < \infty$. Then $|E_1 \setminus E_2| = |E_1| - |E_2|$.

Pf: $E_1 = E_2 \cup (E_1 \setminus E_2)$ is union of disjoint m'ble sets, so $|E_1| = |E_2| + |E_1 \setminus E_2|$. $\square$

- Notation: $E_k \nearrow E$ if $E_1 \subseteq E_2 \subseteq \ldots$ and $E = \bigcup E_k$

$E_k \searrow E$ if $E_1 \supseteq E_2 \supseteq \ldots$ and $E = \bigcap E_k$

- Thm 3.26: Continuity of Lebesgue measure. Let $\{E_k\}$ be m'ble.

i) if $E_k \nearrow E$, then $\lim_{k\to\infty} |E_k| = |E|$

ii) if $E_k \searrow E$ and some $|E_j| < \infty$, then $\lim_{k\to\infty} |E_k| = E$

Proof: (i): If some $|E_k| = \infty$, then all larger has ∞ measure so (i) holds. So assume $|E_k| < \infty$ $\forall k$. We write $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \ldots$, this is a disjoint union of m'ble sets. By countable additivity,
$|E| = |E_1| + \sum_{k=1}^{\infty} |E_{k+1} \setminus E_k| = |E_1| + \sum_{k=1}^{\infty} (|E_{k+1}| - |E_k|)$
$= |E_1| + \lim_{n\to\infty} \sum_{k=1}^{n-1} (|E_{k+1}| - |E_k|) = |E_1| + \lim_{n\to\infty} (|E_n| - |E_1|) = \lim_{n\to\infty} |E_n|$.
(ii): Assume $|E_1| < \infty$. We write $E_1 = E \cup (E_1 \setminus E_2) \cup (E_2 \setminus E_3) \cup \ldots$, this is a disjoint union of m'ble sets. By countable additivity,
$|E_1| = |E| + \sum_{k\in\mathbb{N}} (|E_k| - |E_{k+1}|) = |E| + \lim_{n\to\infty} \sum_{k=1}^{n-1} (|E_k| - |E_{k+1}|) = |E| + \lim_{n\to\infty} (|E_1| - |E_n|)$
$\Rightarrow |E| = \lim_{n\to\infty} |E_n|$. $\square$

↳ (ii) counterexample: $E_k = \mathbb{R} \setminus [-k, k]$

- thm 3.27: Let $\{E_k\}$ be sets in $\mathbb{R}^n$, $E_k \nearrow E$. Then $\lim_{k\to\infty} |E_k|_e = |E|_e$

Proof: We know $\exists$ a G_δ set H_k st. $H_k \supseteq E_k$ and $|H_k|_e = |E_k|_e$.
Define $V_m = \bigcap_{k=m}^{\infty} H_k$, then $V_1 \subseteq V_2 \subseteq \ldots$ and each is m'ble (and also G_δ).
Let $V = \bigcup_{m=1}^{\infty} V_m$, then $V_m \nearrow V$. By prev thm, $\lim_{m\to\infty} |V_m| = |V|$.
If $k \geq m$, $E_m \subseteq E_k \subseteq H_k$ so $E_m \subseteq \bigcap_{k=m} H_k = V_m \subseteq H_m$
$\Rightarrow |E_m|_e \leq |V_m|_e \leq |H_m|_e = |E_m|_e \Rightarrow |V_m|_e = |E_m|_e$. Then $\lim_{m\to\infty} |V_m| = \lim_{m\to\infty} |E_m|_e$
$= |V| = |\bigcup_{m\in\mathbb{N}} V_m| \geq |\bigcup_{m\in\mathbb{N}} E_m|_e = |E|$. And $\lim_{m\to\infty} |E_m|_e \leq |E|$ by monotonicity ($E_m \subseteq E$). $\square$

- thm 3.28: E is m'ble iff it differs from a G_δ or an F_σ by a set of measure 0.
  - (a): E m'ble iff $E = H \setminus Z$ with H a G_δ and $|Z|_e = 0$
  - (b): E m'ble iff $E = F \cup Z$ with F a F_σ and $|Z|_e = 0$

Proof: (a): ($\Leftarrow$): Suppose $E = H \setminus Z$ w/ H a G_δ and $|Z|_e = 0$, then H m'ble and Z m'ble. Hence E m'ble.
($\Rightarrow$): Suppose E m'ble. Choose open G_k s.t. $|G_k \setminus E| < \frac{1}{k}$. Let $H = \bigcap G_k$, this is a G_δ. Let $Z = H \setminus E$, we must show $|Z|_e = 0$. $Z = H \setminus E = (\bigcap G_k \setminus E)$
$\Rightarrow |Z|_e \leq |G_k \setminus E| < \frac{1}{k}$ for all k so $|Z|_e = 0$.
(b): ($\Leftarrow$): Suppose $E = F \cup Z$, then F, Z are m'ble hence E m'ble.
($\Rightarrow$): Suppose E m'ble. Then E^c m'ble, and by (a), $E^c = H \setminus Z$ where H is G_δ and $|Z|_e = 0$. Then $H = \bigcap G_k$, G_k open. Thus $E = \bigcup_{k\in\mathbb{N}} (G_k^c) \cup Z$ and we're done. $\square$

- thm 3.29: Suppose $|E|_e < \infty$, then E is m'ble iff $\forall \varepsilon > 0$ we can write $E = (S \cup N_1) \setminus N_2$ where S is a finite union of nonoverlapping intervals and $|N_1|_e < \varepsilon$, $|N_2|_e < \varepsilon$ Proof: HW
- thm 3.30: Carathéodory's Characterization: E is m'ble iff $\forall A \subseteq \mathbb{R}^n$, $|A|_e = |A \cap E|_e + |A \cap E^c|_e$

Proof: ($\Rightarrow$): Suppose E m'ble, let $A \subseteq \mathbb{R}^n$, choose a G_δ H with $|H|_e = |A|_e$.
$H = (H \cap E) \cup (H \cap E^c)$ (disjoint) $\Rightarrow |H| = |H \cap E| + |H \cap E^c|$ and $|H| = |A|_e$,
also $A \subseteq H$ so $|A|_e = |H| = |H \cap E| + |H \cap E^c| \geq |A \cap E| + |A \cap E^c|$. By subadditivity, since $A = (A \cap E) \cup (A \cap E^c)$, $|A| \leq |A \cap E| + |A \cap E^c|$.
($\Leftarrow$): Assume $|E| < \infty$, Choose a G_δ $H \supseteq E$ s.t. $|H|_e = |E|_e$. By assumption, any $A = H$,
$|H|_e = |H \cap E|_e + |H \cap E^c|_e \Rightarrow |E|_e = |E|_e + |H \cap E^c| \Rightarrow |H \setminus E|_e = 0$.
So let $Z = H \setminus E$ and by thm 3.28, E is m'ble.
For $|E| = \infty$, split into increasing size balls (bounded).

- thm 3.32: Let $E \subseteq \mathbb{R}^n$. $\exists$ G_δ $H \supseteq E$ s.t. $|E \cap M|_e = |H \cap M|$ $\forall M$ m'ble.

Pf: First Assume $|E|_e < \infty$, then $\exists G_\delta$ set $H \supseteq E$ st $|H| = |E|_e$. By Carathéodory,
$|E|_e = |E \cap M|_e + |E \cap M^c|_e$. Next, $H \cap M$ and $H \cap M^c$ are disjoint and m'ble so

$$\left.\begin{array}{l} |H| = |H \cap M| + |H \cap M^c| \\ \quad \shortparallel \\ |E|_e \geq |E \cap M|_e + |E \cap M^c|_e \end{array}\right\} \Rightarrow |H \cap M| \leq |E|_e - |E \cap M^c| = |E \cap M|_e \quad \text{so } |H \cap M| \leq |E \cap M|_e$$

By monotonicity, as $H \supseteq E$, $|H \cap M| \geq |E \cap M|_e$. $\square$

- def: let $T: \mathbb{R}^n \to \mathbb{R}^n$. We say T is c-Lipschitz if $\forall x, y \in \mathbb{R}^n$, $|T(x) - T(y)| \leq c|x-y|$.
- thm 3.33: If T is a Lipschitz transformation and $E \subseteq \mathbb{R}^n$ is m'ble, then TE is m'ble.

Proof: Claim 1: If F is a F_σ, then TF is F_σ and hence m'ble.
Note that T is continuous, so if K compact then TK compact. If F is F_σ
$F = \bigcup_{k \in \mathbb{N}} E_k$ where each E_k compact (every closed set is a count. union of compact)
$TF = \bigcup_{k \in \mathbb{N}} TE_k$ and each TE_k is compact, so TF is F_σ.
Claim 2: If $|Z| = 0$, then $|TZ| = 0$ [T can't stretch sets too much]
For $E \subseteq \mathbb{R}^n$, define $\operatorname{diam}(E) := \sup\{|x-y| : x, y \in E\}$. Then
$\operatorname{diam}(TE) = \sup\{|Tx - Ty| : x, y \in E\} \leq \sup\{c|x-y| : x, y \in E\} = c \cdot \operatorname{diam}(E)$.
Let I be a cube in $\mathbb{R}^n$, and each side has length ℓ. Then $|I| = \ell^n$

Also $\ell \le \operatorname{diam}(I) \le n\ell$. Then $\operatorname{diam}(TI) \le c \cdot \operatorname{diam}(I) \le cn\ell$

So TI is contained in a cube of sides $2cn\ell$. Then

$|TI| \le (2cn\ell)^n = c_1 \ell^n = c_1 |I|$ (where $c_1 = (2cn)^n$.

Let $|Z|=0$. Let $\varepsilon > 0$. We claim we can find cubes $\{I_k\}$ s.t.

$Z \subseteq \bigcup_k I_k$ and $\sum_k |I_k| < \varepsilon/c_1$ (we can do it for intervals, cover each interval

w/ small cubes). By subadditivity, $|TZ|_e \le \sum_k |TI_k|_e \le c_1 \sum_k |I_k|$

$< c_1(\varepsilon/c_1) = \varepsilon$, so $|TZ|_e = 0$.

Claim 3: If E m'ble, then TE m'ble

We know $E = F \cup Z$ for F, F_σ and $|Z|=0$. Then $TE = TF \cup TZ$

and both are m'ble, so TE is m'ble. □

- Axiom of Choice: Let A be an index set and $\{E_\alpha : \alpha \in A\}$ be disjoint sets indexed by A. Then $\exists$ a set consisting of exactly one element from each E_α.
- Lemma 3.37: Let $E \subseteq \mathbb{R}$ be m'ble and $|E| > 0$. Let $D = \{x-y : x,y \in E\}$ (the difference set of E). Then D contains an interval centered on the origin.

Proof: Let $\varepsilon > 0$. $\exists$ open $G \supseteq E$ s.t. $|G| < (1+\varepsilon)|E|$ (thm 3.6).

$G = \bigcup_{k\in\mathbb{N}} I_k$ where I_k are nonoverlapping closed intervals. Let $E_k = I_k \cap E$,

then E_k is m'ble and as $E \subseteq G$, $E = \bigcup_{k\in\mathbb{N}} E_k$. As the $\{I_k\}$ are

nonoverlapping, one-dim intervals, $I_j \cap I_k$ is $\emptyset$ or one point.

So $E_j \cap E_k$ is either $\emptyset$ or a single pt, let S be the set

of these singletons. S is countable so $|S|=0$. Then

$E \setminus S$ is the union of disjoint m'ble sets, $\{E_k \setminus S\}_{k\ge1}$. So

$|E\setminus S| = \sum_{k\in\mathbb{N}} |E_k \setminus S|$, and $|E \setminus S| = |E \setminus (E\cap S)| = |E| - |E \cap S| = |E|$ (with $|E\cap S| = 0$)

Similarly, $|E_k \setminus S| = |E_k|$, hence $|E| = \sum_{k\in\mathbb{N}} |E_k|$. Also, $|G| = \sum |I_k|$ b/c

they are nonoverlapping. So $\sum_{k\in\mathbb{N}} |I_k| \le |G| < (1+\varepsilon)|E| = (1+\varepsilon) \sum_{k\in\mathbb{N}} |E_k|$.

Since $I_k \supseteq E_k$, for at least one $k = k_0$, $|I_{k_0}| < (1+\varepsilon)|E_{k_0}|$.

Let $\varepsilon = 1/3$, $I = I_{k_0}$, $\mathcal{E} = E_{k_0}$. We know $|I| < (1+\varepsilon)|E_{k_0}| = \frac{4}{3}|\mathcal{E}|$.

We claim if $d \in \mathbb{R}$ and $|d| < \frac{1}{2}|I|$ then $\mathcal{E}_d = \mathcal{E} + d = \{x + d : x \in \mathcal{E}\}$

must intersect $\mathcal{E}$. $\mathcal{E} + \mathcal{E}_d$ is contained in an interval of at max

Suppose FSOC it did not

length $|I|+|d|$. $\mathcal{E}$ and $\mathcal{E}_d$ are disjoint, m'ble, so
$$\underbrace{|\mathcal{E}|+|\mathcal{E}_d|}_{=2|\mathcal{E}|} = |\mathcal{E}\cup\mathcal{E}_d| \leq |I|+|d| < \tfrac{3}{2}|I|$$
$2|\mathcal{E}| \Rightarrow |\mathcal{E}| < \frac{3}{4}|I|$ ※. So if $|d| < \frac{1}{2}|I|$ then $\mathcal{E}\cap\mathcal{E}_d \neq \emptyset$.

For $|d| < \frac{1}{2}|I|$, we have $\mathcal{E}\cap\mathcal{E}_d \neq \emptyset$. So $\exists x, y \in \mathcal{E}$ with $y+d \in \mathcal{E}_d$ and $x+y \in \mathcal{E}\cap\mathcal{E}_d \Rightarrow d = x-y \in D$, hence $(-\frac{1}{2}|I|, \frac{1}{2}|I|) \subseteq D$. □

- thm 3.38: Assume AoC. Then $\exists$ a nonmeasurable set.

Proof: Define an equiv relation $x \sim y \iff x-y \in \mathbb{Q}$. The equiv classes have the form $E_x = \{x+r : r \in \mathbb{Q}\} = x+\mathbb{Q}$. If $x, y \in \mathbb{R}$, then either $E_x = E_y$ (if $x-y \in \mathbb{Q}$) or $E_x \cap E_y = \emptyset$ (if $x-y \notin \mathbb{Q}$). Now choose one rep from each equiv class E_x, call it E. If $x, y \in E$ and $x \neq y$, then they belong to diff equiv classes. So $x-y \notin \mathbb{Q}$. So the difference set $D = \{x-y : x-y \in E\}$ cannot contain any interval. By the prev lemma, if E m'ble, then $|E|=0$. Suppose this was true. But $\mathbb{R} = \bigcup_{r\in\mathbb{Q}} \{x+r : x \in E\}$ is the countable union. By subadditivity,
$= \bigcup_{r\in\mathbb{Q}} (r+E)$ $\qquad \infty = |\mathbb{R}| \leq \sum_{r\in\mathbb{Q}} |r+E| = 0$ ※.

So E cannot be m'ble. □

- let $E \subseteq \mathbb{R}^n$, $f: E \to \mathbb{R}\cup\{-\infty, \infty\}$.
  - f is (Lebesgue) measurable if $\forall a \in \mathbb{R}$, $\underbrace{\{x \in E : f(x) > a\}}_{=\{f>a\}}$ is m'ble
  - $E = \{f=-\infty\} \cup \bigcup_{k=1}^{\infty} \{f > -k\} \to$ ★ we'll always assume $\{f=-\infty\}$ is m'ble
- let E be open and $f: E \to \mathbb{R}$ be continuous. Then $\{f>a\} = f^{-1}(a,\infty)$ is open for all $a \in \mathbb{R}$, thus m'ble and f is m'ble.
- let $E \subseteq \mathbb{R}^n$ and $\chi_E(x) = \begin{cases} 1 & x\in E \\ 0 & x\notin E \end{cases}$ be its characteristic function. $\{\chi_E > a\} = \begin{cases} \mathbb{R}^n & \text{if } a<0 \\ E & \text{if } 0\leq a<1 \\ \emptyset & \text{if } a\geq 1 \end{cases}$
  So χ_E is m'ble iff E is m'ble.
- let $E \subseteq \mathbb{R}^n$ be Borel and $f: E \to \mathbb{R}\cup\{-\infty,\infty\}$ if $\forall a \in \mathbb{R}$, $\{f>a\}$ is Borel.
- thm 4.1 $E \subseteq \mathbb{R}^n$ m'ble and $f: E \to \mathbb{R}$. Then f is m'ble iff $\forall a \in \mathbb{R}$, any of the following are m'ble: $\{f>a\}$, $\{f\geq a\}$, $\{f<a\}$, $\{f\leq a\}$

Proof: $(1\Rightarrow 2)$: $\{f \geq a\} = \bigcap_{k=1}^{\infty} \{f > a - \frac{1}{k}\}$

$(2\Rightarrow 3)$: $\{f<a\} = E \setminus \{f \geq a\}$

$(3\Rightarrow 4)$: $\{f \leq a\} = \bigcap_{k=1}^{\infty} \{f < a + \frac{1}{k}\}$

$(4\Rightarrow 1)$: $\{f>a\} = E \setminus \{f \leq a\}$ □

- If f m'ble, $\forall a \in \mathbb{R}$, $\{f=a\} = \{f \geq a\} \cap \{f \leq a\}$ is m'ble, but the converse is not true.
- Cor 4.2: let E m'ble and $f: E \to \mathbb{R} \cup \{-\infty, \infty\}$ m'ble.
  $\{f > -\infty\}$, $\{f < \infty\}$, $\{f = \infty\}$, $\{a \leq f < b\}$, $\{a < f < b\}$, $\{f = a\}$ are m'ble.
  ↳ f m'ble iff $\{a < f < \infty\}$ m'ble $\forall a \in \mathbb{R}$.
- thm 4.3: let $E \subseteq \mathbb{R}^n$ m'ble and $f: E \to \mathbb{R}$. Then f m'ble iff $\forall$ open $G \subseteq \mathbb{R}$, $f^{-1}(G)$ is m'ble.
  Proof: ($\Leftarrow$): let $a \in \mathbb{R}$ and $G = (a, \infty)$, which is open. Hence $f^{-1}(G) = \{f > a\}$ is m'ble.
  ($\Rightarrow$): Let G be open. $G = \bigcup_{k=1}^{\infty} (a_k, b_k)$ (can write as disjoint, but not needed)
  Then $f^{-1}(G) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, b_k) = \bigcup_{k=1}^{\infty} \{a_k < f < b_k\}$ is m'ble. □
- thm 4.4: let $A \subseteq \mathbb{R}$ dense. Then f m'ble iff $\{f > a\}$ is m'ble $\forall a \in A$.
- def: let $E \subseteq \mathbb{R}^n$. A property holds almost everywhere (a.e.) in E if it holds everywhere in E except on a set of measure 0.
- thm 4.5: let f m'ble and $g = f$ a.e. Then g m'ble and $\forall a$, $|\{g > a\}| = |\{f > a\}|$
  Proof: Let $Z = \{f \neq g\}$, and we know $|Z| = 0$. let $a \in \mathbb{R}$. Then
  $\{g > a\} = (\{g > a\} \cap Z) \cup (\{g > a\} \setminus Z) = \underbrace{(\{g > a\} \cap Z)}_{\text{measure } 0} \cup \underbrace{(\{f > a\} \setminus Z)}_{\text{m'ble}}$
  So $\{g > a\}$ is m'ble, and g m'ble.
  By subadditivity, $|\{g > a\}| \leq 0 + |\{f > a\}|$. Reversing f and g gives $|\{f > a\}| \leq 0 + |\{g > a\}|$, so $=$ holds. □
- def: $E \subseteq \mathbb{R}^n$ and f is defined a.e. in E, so $\exists Z$ w/ $|Z| = 0$ and f is defined on $E \setminus Z$. Then f is m'ble in E if it is m'ble in $E \setminus Z$.
- composition of m'ble fns is not necessarily m'ble.
- comp. of non m'ble and m'ble fn can be m'ble.
  ex: let $E \subseteq \mathbb{R}$ be not m'ble. define $f(x) = \begin{cases} 1 & x \in E \\ -1 & x \notin E \end{cases}$, which is not m'ble.
  $\phi(x) = x^2$ is cont, hence m'ble, but $\phi(f(x))$ is m'ble.
- $E \subseteq \mathbb{R}^n$ Let $\phi: \mathbb{R} \to \mathbb{R}$ be cont and $f: E \to \mathbb{R} \cup \{-\infty, \infty\}$ be finite a.e and m'ble. Then $\phi \circ f$ m'ble.
  ↳ $\phi \circ f$ is not defined on a set of measure 0.
  Pf: $Z = \{f = \infty\} \cup \{f = -\infty\}$ has measure 0, f is finite everywhere on $E \setminus Z$.
  let G be open. Then $(\phi \circ f)^{-1}(G) = \{x \in E : \phi(f(x)) \in G\} = \{x \in E : f(x) \in \phi^{-1}(G)\}$

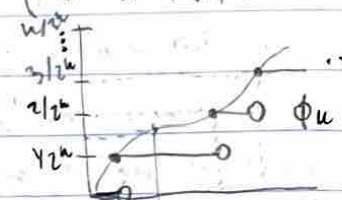
$= f^{-1}(\phi^{-1}(G))$. Note $\phi^{-1}(G)$ is open in $\mathbb{R}$. As f is m'ble, $f^{-1}(\phi^{-1}(G))$ is m'ble.
So $(\phi \circ f)^{-1}$ is m'ble (thm 4.3). $\square$

- thm 4.7: If f, g m'ble, then $\{f > g\}$ is m'ble.
Proof: Let $\mathbb{Q} = \{r_1, \dots\}$. Then $\{f > g\} = \bigcup_{k=1}^{\infty} (\{f > r_k\} \cap \{g < r_k\})$ is m'ble
(note $f(x) > g(x)$ iff $\exists r_k$ s.t. $f(x) > r_k > g(x)$) $\square$
- thm 4.8: Let $E \subseteq \mathbb{R}^n$ be m'ble and $f: E \to \mathbb{R}$. If f is m'ble and $\lambda \in \mathbb{R}$, then λf and $f + \lambda$ are m'ble.
Proof: Let $a \in \mathbb{R}$. $\{f + \lambda > a\} = \{f > a - \lambda\}$ is m'ble.
Assume $\lambda > 0$, then $\{\lambda f > a\} = \{f > a/\lambda\}$ is m'ble
If $\lambda < 0$, then $\{\lambda f > a\} = \{f < a/\lambda\}$ is m'ble.
If $\lambda = 0$, then $\{\lambda f > a\} = \begin{cases} E & \text{if } a < 0 \\ \emptyset & \text{if } a \geq 0 \end{cases}$ is m'ble. $\square$
- thm 4.9: Let f, g m'ble and finite a.e. Then $f+g$ is m'ble.
Proof: We can assume f and g are finite everywhere. Let $a \in \mathbb{R}$.
Then $a - g = a + (-1)g$ is m'ble (by above). Then $\{f+g > a\} = \{f > a - g\}$ is m'ble by thm 4.7. $\square$
  - finite linear combination of m'ble fns is m'ble.
- Prop: Let $E \subseteq \mathbb{R}^n$ m'ble and let $E_1 \subseteq E$ m'ble. Let $f: E \to \mathbb{R}$ be m'ble, then $f|_{E_1}$ is m'ble.
Proof: $\{f|_{E_1} > a\} = \{f > a\} \cap E_1$ is m'ble. $\square$
- thm 4.10: Let f, g be m'ble.
  a) fg is also m'ble b) If $g \neq 0$, f/g is m'ble.
- thm 4.11: Let $\{f_k\}$ be a sequence of measurable functions. Then $\sup_k f_k$, $\inf_k f_k$ are m'ble
Proof: If $a \in \mathbb{R}$, $\{\sup f_k > a\} = \bigcup_k \{f_k > a\}$ is m'ble. Next, $\inf f_k = -\sup(-f_k)$ is m'ble
or, $\{\inf_k f_k > a\} = \bigcap_k \{f_k > a\}$ is m'ble. $\square$
- thm 4.12: Let $\{f_k\}$ be m'ble.
  a) $f = \limsup f_k$, $h = \liminf f_k$ are m'ble b) If $\lim f_k$ exists, then it's m'ble.
Proof: $\limsup f_k = \inf_{j \geq 1} (\sup_{k \geq j} f_k)$ and $\liminf f_k = \sup_{j \geq 1} (\inf_{k \geq j} f_k)$ are m'ble.
If $\lim f_k$ exists, then $\lim f_k = \limsup f_k$ is m'ble. $\square$

- def: let $\{E_j\}_{j=1}^{N}$ be disjoint sets in $\mathbb{R}^n$, and $\{a_j\}_{j=1}^{N}$ be distinct reals. Then $f = \sum_{j=1}^{N} a_j \chi_{E_j}$ is a simple function. (fns that take finitely many values)
  ↳ f m'ble iff each E_j is m'ble.
- thm 4.13: let $E \subseteq \mathbb{R}^n$. ① let $f : E \to \mathbb{R} \cup \{-\infty, \infty\}$. We can write $f(x) = \lim_{k\to\infty} \phi_k(x)$ $\forall x$, where $\{\phi_k\}$ are simple fns. ② let $f : E \to [0, \infty)$. Then $\exists$ simple fns $0 \le \phi_1 \le \phi_2 \le \dots \le f$ s.t. $\lim_{k\to\infty} \phi_k(x) = f(x)$ $\forall x$. ③ If f m'ble, we can construct it so that ϕ_k's are all m'ble.

<u>Proof (2)</u>: let $f : E \to [0, \infty)$. Define for each $k \ge 1$, $\phi_k(x) = \begin{cases} \frac{j-1}{2^k} & \text{if } \frac{j-1}{2^k} \le f(x) < \frac{j}{2^k} \text{ for some } 1 \le j \le k2^k \\ k & \text{if } f(x) \ge k \end{cases}$

Each ϕ_k takes only finitely many values, so it is simple.



(monotonicity of ϕ_k's): Fix $x \in E$ and first assume $f(x) < \infty$. Then for large enough k, $f(x) < k$. Then for some $1 \le j \le k2^k$, $\frac{j-1}{2^k} \le f(x) < \frac{j}{2^k}$, so $\phi_k(x) = \frac{j-1}{2^k}$, so $0 \le \phi_k(x) \le f(x)$. Now consider $\phi_{k+1}(x)$, note $\frac{2j-2}{2^{k+1}} \le f(x) < \frac{2j}{2^{k+1}}$. either way $\phi_{k+1}(x) \ge \phi_k(x)$.
If $f(x) = \infty$, then $f(x) \ge k$ $\forall k$, hence $\phi_k(x) = k$ and $\phi_1(x) \le \phi_2(x) \le \dots$
(convergence): As $\{\phi_k\}$ is increasing, $\lim_{k\to\infty} \phi_k(x)$ exists and is either finite or infinite.
Fix $x \in E$. First if $f(x) < \infty$, for large enough k, $f(x) \in [0, k]$. Then choose j satisfying $\frac{j-1}{2^k} \le f(x) < \frac{j}{2^k}$, but then $0 \le f(x) - \phi_k(x) \le \frac{j}{2^k} - \frac{j-1}{2^k} = \frac{1}{2^k} \to 0$ as $k \to \infty$.
Second if $f(x) = \infty$, $\phi_k(x) = k$ and $\lim_{k\to\infty} \phi_k(x) = \infty$. Thus $f(x) = \lim_{k\to\infty} \phi_k(x)$. □

<u>Proof (1)</u>: Now let $f : E \to \mathbb{R} \cup \{-\infty, \infty\}$. Define $f^+ = \max\{0, f\}$, $f^- = -\min\{0, f\}$. are nonnegative on E and $f = f^+ - f^-$. By (2), $\exists$ simple fns $\{\phi_k^+\}$ and $\{\phi_k^-\}$ that increase to f^+ and f^-, respectively. Define $\phi_k = \phi_k^+ - \phi_k^-$, which is also simple as each ϕ_k^+, ϕ_k^- take finitely many values. For some $x \in E$, if $f(x) < \infty$, then $f^+(x), f^-(x) < \infty$ also, so $\lim_{k\to\infty} \phi_k(x) = \lim_{k\to\infty} (\phi_k^+(x) - \phi_k^-(x)) = f^+(x) - f^-(x) = f(x)$.
If $f(x) = \infty$ or $-\infty$, then at most one of f^+, f^- is infinite. Again, $\lim_{k\to\infty} \phi_k(x) = f(x)$. □

<u>Proof (3)</u>: Assume f is m'ble, first that $f \ge 0$. Then in (1),
$\phi_k(x) = \sum_{j=1}^{k2^k} \frac{j-1}{2^k} \chi_{\{\frac{j-1}{2^k} \le f < \frac{j}{2^k}\}} + k \chi_{\{f \ge k\}}$ is m'ble. □

- <u>def</u>: let $f : E \to \mathbb{R} \cup \{-\infty, \infty\}$. Let $x_0 \in E$ be a limit pt of E.
  (a) We say f is upper semicont (u.s.c) @ x_0 if $\limsup_{x \to x_0, x \in E} f(x) \le f(x_0)$

  (b) We say f is lower semicont (lsc) @ x_0 if $\liminf_{x \to x_0, x \in E} f(x) \ge f(x_0)$

- Remarks: if $f(x_0) = \infty$, f is usc @ x_0. If $f(x_0) = -\infty$, f is lsc @ x_0.
- Sps $f(x_0)$ is finite. Then f is cont iff f is lsc and usc @ x_0.
- ex. Let $f = \chi_{[0,\infty)}$. $\liminf_{x \to 0} f = 0 < 1 = f(0)$ so f is not lsc @ 0
  $\limsup_{x \to 0} f = 1 = f(0)$ so f is usc @ 0.

- (ε-δ for usc, lsc): Let $f: E \to \mathbb{R} \cup \{-\infty, \infty\}$. Let $x_0 \in E$ be a limpt. of E w/ $f(x_0)$ finite.
  a) f is usc @ x_0 iff $\forall M > f(x_0)$, $\exists \delta > 0$ st $|x - x_0| < \delta$ and $x \in E \Rightarrow f(x) < M$
  b) f is lsc @ x_0 iff $\forall M < f(x_0)$, $\exists \delta > 0$ st $|x - x_0| < \delta$ and $x \in E \Rightarrow f(x) > M$
- Lemma: Let $f, g : E \to \mathbb{R} \cup \{-\infty, \infty\}$. Assume $x_0 \in E$ is a limpt of E. Assume f, g are usc at x_0. Then $f+g$ is usc and if $f, g \geq 0$ then fg is usc.
- Lemma: Let $f_k : E \to \mathbb{R} \cup \{-\infty, \infty\}$ and let $x_0 \in E$ be a limpt of E. Assume f_k's are usc @ x_0. Then (a) $\inf_{k \geq 1} f_k$ is usc @ x_0. (b) If $f_1 \geq f_2 \geq \dots$ and $f = \lim f_k$ exists, then f usc @ x_0.

Proof: (a) Let $j \geq 1$. Then $\inf_{k \geq 1} f_k \leq f_j$. So $\underbrace{\limsup_{x \to x_0, x \in E} \left(\inf_{k \geq 1} f_k \right)}_{} \leq \limsup_{x \to x_0, x \in E} (f_j) \leq f_j(x_0)$ (usc)
Taking inf over j, since LHS is indep of j, $\leq \inf_{j \geq 1} f_j(x_0)$, so $\inf_{k \geq 1} f_k$ is lsc @ x_0.
(b) Since $\{f_k\}$ decreasing, $f = \lim_{k \to \infty} f_k = \inf_{k \geq 1} f_k$ so f is lsc @ x_0 by (a). □

- def: Let $f: E \to \mathbb{R} \cup \{-\infty, \infty\}$.
  a) f is usc relative to E if f is usc at every limit point $x_0 \in E$.
  b) f is lsc relative to E if f is lsc at every limit point $x_0 \in E$.
- def: Let $E \subseteq \mathbb{R}^n$, $H \subseteq E$
  a) H is relatively closed (to E) if $H = E \cap F$ for some closed F
  b) H is relatively open (to E) if $H = E \cap G$ for some open G.
  * H is relatively closed iff $\forall \{x_k\}$ in H with limit point $x_0 \in E$, then $x_0 \in H$.
- Thm 4.14: Let $f: E \to \mathbb{R} \cup \{-\infty, \infty\}$
  a) f is usc relative to E iff $\{f \geq a\}$ is rel. closed $\forall a \in \mathbb{R}$
  b) f is lsc relative to E iff $\{f \leq a\}$ is rel. closed $\forall a \in \mathbb{R}$

Proof: (a): ($\Rightarrow$): Let $\{x_k\}$ be a sequence in $\{f \geq a\}$ w/ limpt $x_0 \in E$, with $x_k \neq x_0$. By usc, $f(x_0) \geq \limsup_{k \to \infty} f(x_k) \geq a$, so $x_0 \in \{f \geq a\}$ and $\{f \geq a\}$ rel closed.
($\Leftarrow$): AFSOC that $\exists$ a limit point $x_0 \in E$ that f is not usc at. Then $\exists M > f(x_0)$ and a sequence $\{x_k\}$ in E with $\lim_{k \to \infty} x_k = x_0$ but $f(x_k) \geq M$ for $k \geq 1$. Then for $k \geq 1$, $x_k \in \{f \geq M\} \cap E$, but $f(x_0) < M$ so $x_0 \notin \{f \geq M\}$, contradicting the fact that $\{f \geq M\}$ rel closed.
(b): f lsc to $E \Leftrightarrow -f$ usc to $E \Leftrightarrow \{-f \geq -a\}$ rel closed $\forall a \Leftrightarrow \{f \leq a\}$ rel closed $\forall a$ □

- Remark: Similarly, f usc rel to E $\Leftrightarrow$ $\{f<a\}$ rel open $\forall a$
f lsc to E $\Leftrightarrow$ $\{f>a\}$ rel open $\forall a$
- Corollary 4.15: Let $f: E \to \mathbb{R}$
a) f is cont on E iff $\forall a$, $\{f \geq a\}$, $\{f \leq a\}$ rel closed
b) f is cont in E iff $\forall a$, $\{f<a\}$, $\{f>a\}$ rel open
- Remark: f cont in E $\Rightarrow$ $\{f=a\}$ is rel closed $\forall a \in \mathbb{R}$ (but not $\Leftarrow$)
- Corollary 4.16: Let $E \subseteq \mathbb{R}^n$ be m'ble. If f is usc or lsc to E, then f is m'ble.
Proof: Assume f is usc rel to E. Then $\{f \geq a\}$ is rel closed $\forall a$.
Then $\{f \geq a\} = E \cap F$ when F closed, hence m'ble, so f m'ble.
- Thm 4.17 (Egorov's Thm): Let $E \subseteq \mathbb{R}^n$ m'ble and $|E|$ finite. Let $f_k: E \to \mathbb{R}$ m'ble, and assume $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. $x \in E$. Then given $\varepsilon > 0$, $\exists F \subseteq E$ closed st. $|E \setminus F| < \varepsilon$ and $\{f_k\}$ converges uniformly to f on F, i.e.

f finite a.e.

$\sup_{x\in F} \{|f_k(x) - f(x)| : x \in F\} \to 0$ as $k \to \infty$
- Lemma 4.18: Assume the hypothesis of Egorov's Thm. Let $\varepsilon, \eta > 0$. Then $\exists F \subseteq E$ closed and an integer L st. $|f_k(x) - f(x)| < \varepsilon$ for $x \in F$ and all $k \geq L$, and $|E \setminus F| < \eta$

Proof: Let $Z_1 = \{f = \pm\infty\}$ and $Z_2 = \{x \in E : f_k(x)$ doesn't converge to $f(x)\}$.
By hypothesis, $|Z_1| = |Z_2| = 0$. For $m \geq 1$ let $E_m = \{x \in E \setminus (Z_1 \cup Z_2) :$ for all $k \geq m$, $|f_k(x) - f(x)| < \varepsilon\}$
$= \bigcap_{k\geq m} \{x \in E \setminus (Z_1 \cup Z_2) : |f_k(x) - f(x)| < \varepsilon\} = \bigcap_{k \geq m} (\{|f_k - f| < \varepsilon\} \setminus (Z_1 \cup Z_2))$.
Z_1, Z_2 m'ble. Also $f = \lim_{k\to\infty} f_k(x)$ exists a.e. and $\{f_k\}$ m'ble, so f is m'ble. So $|f_k - f|$ m'ble, hence E_m is m'ble. Also $E_1 \subseteq E_2 \subseteq \dots$ Next if $x \in E \setminus (Z_1 \cup Z_2)$, then $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. So $|f_k(x) - f(x)| < \varepsilon$ for large enough k, hence $x \in E_m$ for E_m large enough. So $E_m \nearrow E \setminus (Z_1 \cup Z_2)$, thus:
$\lim_{m\to\infty} |E_m| = |E \setminus (Z_1 \cup Z_2)| = |E| - |Z_1 \cup Z_2| = |E|$. As $E_m \subseteq E$, we have that $\lim_{m\to\infty} |E \setminus E_m| = \lim_{m\to\infty} (|E| - |E_m|) = 0$. So $\exists$ large enough m_0 st. $|E \setminus E_{m_0}| < \eta/2$.
As E_m m'ble, $\exists$ closed $F \subseteq E_{m_0}$ st $|E_{m_0} \setminus F| < \eta/2$. Then
$|E \setminus F| = |(E \setminus E_{m_0}) \cup (E_{m_0} \setminus F)| < \eta/2 + \eta/2 = \eta$. Also by def of E_{m_0}, as $F \subseteq E_{m_0}$,
$x \in F \Rightarrow x \in E_{m_0} \Rightarrow |f_k(x) - f(x)| < \varepsilon$ for $k \geq m_0$, set $L = m_0$. $\square$

Proof of Egorov: fix $\varepsilon > 0$ and $m \geq 1$. We apply lemma 4.18 with 'η' $= \frac{\varepsilon}{2^m}$ and 'ε' $= \frac{1}{m}$. So $\exists F_m \subseteq E$ closed and integers l_m st. $|E \setminus F_m| < \frac{\varepsilon}{2^m}$, $m \geq 1$ and $\forall x \in F_m$ and $k > l_m$, $|f_k(x) - f(x)| < \frac{1}{m}$. Let $\bigcap_{m \in \mathbb{N}} F_m = F$, which is closed. $|E \setminus F| = |E \setminus \bigcap_{m \in \mathbb{N}} F_m|$ $= |\bigcup_{m \in \mathbb{N}} (E \setminus F_m)| \leq \sum_{m \in \mathbb{N}} |E \setminus F_m| < \sum_{m \in \mathbb{N}} \frac{\varepsilon}{2^m} = \varepsilon$. Secondly, as $F \subseteq F_m$, for $k > l_m$, $\sup\{|f_k(x) - f(x)| : x \in F\} \leq \frac{1}{m} \to 0$ as $m \to \infty$. □

- Remarks a) $|E|$ must be finite. A counterexample, let $E = [0, \infty)$, $f_k(x) = \chi_{[0,k]}$, $f(x) = 1$. Then $\lim_{k \to \infty} f_k(x) = f(x)$ $\forall x \in E$. However $|\{x \in E : |f_k(x) - f(x)| < 1\}| = |(k, \infty)| = \infty$ for $k \geq 1$.
  b) f must be finite a.e. A a counter, let $f_k(x) = k$ for $[0,1] = E$, then $\lim_{k \to \infty} f_k(x) = \infty = f(x)$. But $|\{x \in E : |f_k(x) - f(x)| \geq 1\}| = |[0,1]| = 1$, for $k \geq 1$.
- def: let $E \subseteq \mathbb{R}^n$ and $f : E \to \overline{\mathbb{R}}$ let $A \subseteq E$
  a) let $x_0 \in A$, we say f is continuous at x_0 relative to A if $f(x_0)$ is finite and either
  i) x_0 is an isolated point ii) x_0 is a limpt of A and $\lim_{x \to x_0, x \in A} f(x) = f(x_0)$
  b) We say f is continuous relative to A if it is continuous $\forall x_0 \in A$, relative to a
  $\hookrightarrow \Leftrightarrow$ $f|A$ is continuous
- def: let $E \subseteq \mathbb{R}^n$ m'ble and $f : E \to \overline{\mathbb{R}}$. f has Property C if $\forall \varepsilon > 0$, $\exists F \subseteq E$ closed s.t. $|E \setminus F| < \varepsilon$ and f is continuous relative to F.
- thm 4.20 (Lusin's Thm): Let $f : E \to \mathbb{R}$ be finite on E. Then f m'ble iff f has Property C
- lemma 4.19: Let ϕ be a m'ble simple fn on a m'ble set E. Then ϕ has Property C.

Proof: We can write $\phi = \sum_{k=1}^{N} a_k \chi_{E_j}$ and a_k distinct, E_j disjoint m'ble.
Let $\varepsilon > 0$. Since each E_j m'ble, $\exists F_j \subseteq E_j$ closed st. $|E_j \setminus F_j| < \frac{\varepsilon}{N}$. Let $F = \bigcup_{j=1}^{N} F_j$, which is closed. Then $|E \setminus F| = |\bigcup_{j=1}^{N} E_j \setminus \bigcup_{j=1}^{N} F_j| \leq |\bigcup_{j=1}^{N} (E_j \setminus F_j)| \leq \sum_{j=1}^{N} |E_j \setminus F_j| < \varepsilon$.
Case 1: E is bounded. Then each F_j is bounded and disjoint, so there is pairwise positive distance between every pair F_j. ϕ is constant on each F_j, so continuous when restricted to F_j. Hence ϕ also cont when restricted to F. So ϕ has Property C.
Case 2: E is unbounded. For $m \geq 1$, $cl(B_m(0)) \cap F$ is compact, consisting of disjoint sets $\{F_j \cap cl(B_m(0))\}$, and Case 1 shows ϕ is cont. relative to $F \cap cl(B_m(0))$. In particular, ϕ is cont rel. to F at any $x_0 \in F$ with $|x_0| < m$. But any $x_0 \in F$ satisfies this for any large enough m, so ϕ cont rel to F $\forall x_0 \in F$. So ϕ has Property C.

Proof of Lusin: ($\Rightarrow$): let f be m'ble. By thm 4.13, as f m'ble, $\exists$ m'ble simple fns $\{\phi_k\}$ st. $\lim_{k\to\infty} \phi_k(x) = f(x)$ $\forall x \in E$. By Lemma 4.19, each

(case 1: $|E|<\infty$) ϕ_k has property C. Let $\varepsilon>0$. For each $k\geq 1$, $\exists F_k \subseteq E$ closed st. $|E\setminus F_k| < \frac{\varepsilon}{2^{k+1}}$ and $\phi_k|_{F_k}$ is continuous. By Egorov's Thm, $\exists F_0 \subseteq E$ closed st. $|E\setminus F_0| < \frac{\varepsilon}{2}$ and $\{\phi_k\}$ converge uniformly on F_0 to f.

Let $F = \bigcap_{k\in\mathbb{N}^0} F_k$, which is closed. Then $|E\setminus F| = |E \setminus \bigcap_{k=0}^{\infty} F_k|$ $= |\bigcup_{k=1}^{\infty} (E\setminus F_k)| \leq \sum_{k=0}^{\infty} |E\setminus F_k| < \varepsilon$. Further, since $F \subseteq F_0$, $\{\phi_k\}$ converge uniformly to f on F. Since $\phi_k|_F$ is cont, $f|_F$ is cont (only since ϕ_k converges uniformly), so f has property C.

Case 2: $|E|=\infty$. Let $E_k = E\cap (B_k(0)\setminus B_{k-1}(0))$. As each $|E_k|<\infty$, Case 1 guarantees a closed set $F_k \subseteq E_k$ st. $|F_k\setminus E_k| < \frac{\varepsilon}{2^k}$ and $f|_{F_k}$ cont. Let $F = \bigcup F_k$, then $|E\setminus F| \leq \ldots < \varepsilon$. Also F contains its limpts hence is closed ($\bigcup_{k=1}^{N} F_k$ is closed). As F_j compact disjoint, continuity of $f|_{F_j}$ $\forall j$ gives $f|_F$ continuous

($\Leftarrow$): Suppose f has property C. So for $k\geq 1$, $\exists F_k \subseteq E$ closed st $|E\setminus F_k| < \frac{1}{k}$, and $f|_{F_k}$ continuous. Let $H = \bigcup_{k\in\mathbb{N}} F_k$, an F_σ set.
Let $Z = E\setminus H$, Then $|Z| = |E\setminus \bigcup F_k| \leq |E\setminus F_k| < \frac{1}{k}$ $\forall k$, so $|Z|=0$.
Let $a\in\mathbb{R}$. then $\{f>a\} = (\{f>a\}\cap H) \cup (\{f>a\}\cap H^c)$.
$= (\{f>a\}\cap \bigcup_{k\in\mathbb{N}} F_k) \cup (\{f>a\}\cap Z) = (\bigcup_{k\in\mathbb{N}} (\{f>a\}\cap F_k)) \cup (\{f>a\}\cap Z)$
Since $f|_{F_k}$ cont, so $\{f>a\}\cap F_k$ is rel open (cor 4.15), so m'ble.
Also $|(\{f>a\}\cap Z)| = 0$ so $\{f>a\}\cap Z$ m'ble. Thus $\{f>a\}$ m'ble, so f m'ble. $\square$

- def: Let $E\subseteq \mathbb{R}^n$ m'ble, $f_k : E\to\overline{\mathbb{R}}$, $f: E\to \overline{\mathbb{R}}$ m'ble, finite a.e. We say $\{f_k\}$ converges in measure to f on E if $\forall \varepsilon>0$, $\lim_{k\to\infty} |\{|f-f_k|>\varepsilon\}| = 0$. We write $f_k \xrightarrow{m} f$ on E.

- Remark: Uniform conv $\Rightarrow$ Pointwise conv $\Rightarrow$ Convergence a.e $\Rightarrow$ Convergence in measure (if $|E|<\infty$)

- thm 4.21: Convergence a.e $\Rightarrow$ Convergence in measure.

Proof: Let $E\subseteq \mathbb{R}^n$ and $|E|<\infty$. Let $f, f_k : E\to\overline{\mathbb{R}}$ m'ble, finite a.e.
Assume $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. in E. Let $\varepsilon, \eta>0$, by lemma 4.18, $\exists F\subseteq E$ closed and $l\in\mathbb{N}$ st. $|E\setminus F| < \eta$ and $|f_k(x) - f(x)| \leq \varepsilon$. $\forall x\in F$, $k\geq l$.

Then for $k \geq L$, $\{|f_k - f| > \varepsilon\} \subseteq E \setminus F$ (since $\leq$ holds inside F),
by monotonicity, $|\{|f_k - f| > \varepsilon\}| < \eta$, but $\eta > 0$ was arbitrary, so $f_k \xrightarrow{m} f$ on E □

- note: we need $|E| < \infty$. Let $E = \mathbb{R}$, $f_k = \chi_{[-k,k]}$ $k \geq 1$, $f = 1$ in $\mathbb{R}$, then $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. in $\mathbb{R}$, but for $0 < \varepsilon < 1$, $|\{|f_k - f| > \varepsilon\}| = |(-\infty, -k] \cup [k, \infty)| = \infty$.
- note: converse is not true! ex. Let $E = [0,1]$. Choose $\{I_k\}_{k\in\mathbb{N}}$ in $[0,1]$ so that i) each pt of $[0,1]$ belongs to infinitely many of I_k's and ii) $\lim_{k\to\infty} |I_k| = 0$.
for example $I_1 = [0, \frac{1}{2}]$, $I_2 [\frac{1}{2}, 1]$; $I_3 = [0, \frac{1}{3}]$, $I_4 = [\frac{1}{3}, \frac{2}{3}]$, $I_5 = [\frac{2}{3}, 1]$; $I_6 = [0, \frac{1}{4}]$, ...
Let $f_k = \chi_{I_k}$, and $f = 0$. If $\varepsilon \in (0,1)$, $|\{|f_k - f| > \varepsilon\}| = |\{|f_k| > \varepsilon| = |I_k| \to 0$
and if $\varepsilon \geq 1$, the measure is 0. So $f_k \xrightarrow{m} f$ on $[0,1]$. But $\lim_{k\to\infty} f_k(x)$ DNE for any $x \in [0,1]$.
- thm 4.22: Convergence in measure $\overset{\text{if } |E|<\infty}{\Rightarrow}$ a subsequence converges a.e.

Proof: Let $E \subseteq \mathbb{R}^n$ be m'ble, $f, f_k : E \to \mathbb{R}$ m'ble, finite a.e. Assume $f_k \xrightarrow{m} f$ on E.
By hypothesis, given $\varepsilon, \eta > 0$, $\exists L$ s.t. $\forall k \geq L$, $|\{|f_k - f| > \varepsilon\}| < \eta$. For $j \geq 1$, choose $\varepsilon = \frac{1}{j}$, $\eta = \frac{1}{2^j}$, so we have corresponding L_j. We can assume $L_1 < L_2 < \ldots$ (choose L_1, then L_2 larger, ...). Let $E_j = \{|f_{L_j} - f| > \frac{1}{j}\}$. Note $|E_j| < \frac{1}{2^j}$.
Let $Z = \limsup_{j\to\infty} E_j = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$. For $m \geq 1$, $|Z| \leq |\bigcup_{j=m}^{\infty} E_j| \leq \sum_{j=m}^{\infty} |E_j| < 2^{-m+1} \to 0$
so $|Z| = 0$. If $x \in E \setminus Z$, then $x \notin \bigcup_{j=m}^{\infty} E_j$ for some m. But then $x \notin E_j$ $(j \geq m)$
so $|f_{L_j}(x) - f(x)| < \frac{1}{j}$ $(j \geq m)$, hence $\lim_{j\to\infty} f_{L_j}(x) = f(x)$ on $E \setminus Z$, i.e. a.e. □

- thm 4.23: Let $E \subseteq \mathbb{R}^n$ m'ble, $f_k : E \to \mathbb{R}$ m'ble. Then $\exists$ m'ble fn $f : E \to \mathbb{R}$ s.t. $f_k \xrightarrow{m} f$ on E iff $\forall \varepsilon > 0$, $|\{|f_j - f_k| > \varepsilon\}| \to 0$ as $j, k \to \infty$
- def: Let $E \subseteq \mathbb{R}^n$ m'ble, $f : E \to [0, \infty]$.
  a) The graph of f over E is $\Gamma(f, E) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E, f(x) < \infty\}$
  b) The region under f over E is $R(f, E) = \{(x,y) \in \mathbb{R}^{n+1} : x \in E, \begin{array}{l} 0 \leq y \leq f(x) \text{ if } f(x) < \infty \\ 0 \leq y < \infty \text{ if } f(x) = \infty \end{array}\}$
- def: (The Lebesgue Integral of f) Assume $R(f, E) \subseteq \mathbb{R}^{n+1}$ is m'ble. Define $\int_E f(x)dx = |R(f, E)|_{(n+1)}$.
We say $\int_E f(x)\,dx$ exists. (also write $\int_E f$) (can be infinite)
- thm 5.1: Let $E \subseteq \mathbb{R}^n$ be m'ble, $f : E \to [0, \infty]$. Then $\int_E f$ exists iff f m'ble
- note: f m'ble $\Leftrightarrow$ $R(f, E)$ m'ble
- def (Cylinder sets): Let $E \subseteq \mathbb{R}^n$. For $0 < a < \infty$, define $E_a = E \times [0, a]$. $E_\infty = E \times [0, \infty)$.

- Lemma 5.2: Let $E \subseteq \mathbb{R}^n$ m'ble and $0 \le a \le \infty$. Then E_a is m'ble in $\mathbb{R}^{n+1}$ and $|E_a|_{(n+1)} = |E_n| \cdot a$ (note $0 \cdot \infty = \infty \cdot 0 = 0$)

Proof: Case 1: E is an interval in $\mathbb{R}^n$, $a < \infty$. Then $E_a = E \times [0, a]$ is an interval in $\mathbb{R}^{n+1}$, so $|E_a|_{(n+1)} = |E|_n \cdot a$.

Case 2: E open, $|E| < \infty$, $a < \infty$. Then $E = \bigcup_{k \in \mathbb{N}} J_k$ where J_k's are nonoverlapping intervals in $\mathbb{R}^n$. Then $\{(J_k)_a\}$ are nonoverlapping intervals in $\mathbb{R}^{n+1}$. So,
$|E_a|_{(n+1)} = |\bigcup_{k=1}^{\infty} (J_k)_a|_{(n+1)} = \sum_{k=1}^{\infty} |(J_k)_a|_{(n+1)} = \sum_{k=1}^{\infty} |J_k| \cdot a = |E|_n \cdot a$

Case 3: E is a bounded G_δ set, $a < \infty$. Then $E = \bigcap_{k \in \mathbb{N}} G_k$, G_k's open. We may assume each $|G_k| < \infty$, if not, replace G_k by its intersection w/ an open ball containing E. We can also assume $G_1 \supseteq G_2 \supseteq \ldots$, if not, replace G_k with $G_1 \cap G_2 \cap \ldots \cap G_k$. Then $|G_k| < \infty$ and $G_k \searrow E$, so $\lim_{k\to\infty} |G_k|_{(n)} = |E|_{(n)}$.
But also, $(G_k)_a \searrow E_a$ and $|(G_k)_a| < \infty$, so again, $\lim_{k\to\infty} |(G_k)_a|_{(n+1)} = |E_a|_{(n+1)}$.
By case 2, as G_k's open, $\lim_{k\to\infty} |G_k|_{(n)} a = |E_a|_{(n+1)}$ so $|E_a|_{(n+1)} = |E|_{(n)} \cdot a$

Case 4: E is bounded m'ble set, $a < \infty$. So $\exists G$, G_δ s.t. $E = G \setminus Z$ and $|Z| = 0$.
We may assume $Z \subseteq G$ (if not, replace Z w/ $Z \cap G$). G is bounded b/c E is.
Then $|E|_{(n)} = |G \setminus Z|_{(n)} = |G|_{(n)} - |Z|_{(n)} = |G|_n$. Next $E_a = G_a \setminus Z_a$ and $|Z_a|_{(n+1)} = 0$ (needs some proof). Also $Z_a \subseteq G_a$, then $|E_a|_{(n+1)} = |G_a \setminus Z_a|_{(n+1)} = |G_a|_{(n+1)} = a|G|_{(n)} = a|E|$.

Case 5: E is possibly unbounded, m'ble and $a < \infty$. Let $E^k = E \cap B_k(0)$. Then $E^k \nearrow E$ and $E^k_a \nearrow E_a$. So by Case 4, $|E^k_a|_{(n+1)} = |E^k|_{(n)} a$, then $\lim_{k\to\infty} |E^k_a|_{(n+1)} = a \lim_{k\to\infty} |E^k_a|_{(n)}$
so $|E_a|_{(n+1)} = a|E|_n$.

Case 6: E is possibly unbounded, m'ble, $a = \infty$. Choose an increasing sequence $\{a_k\}$ finite, pos. nums w/ $a_k \nearrow \infty$. Then $E_{a_k} \nearrow E_\infty$. By case 5, $|E_{a_k}|_{(n+1)} = |E|_{(n)} a_k$, then
$\lim_{k\to\infty} |E_{a_k}|_{(n+1)} = \lim_{k\to\infty} |E|_{(n)} a$ so $|E_a|_{(n+1)} = |E|_n \cdot \infty$. $\square$

- Corollary: If $E \subseteq \mathbb{R}^n$ m'ble, for $0 \le a \le b$, then $|E \times [a,b]|_{(n+1)} = |E_n| (b-a)$
- Lemma 5.3: Let $E \subseteq \mathbb{R}^n$ m'ble, $f: E \to [0, \infty]$ m'ble. Then $|\Gamma(f, E)|_{(n+1)} = 0$.

Proof: Case 1: $|E| < \infty$. Let $\varepsilon > 0$. Divide the y-axis into intervals $[k\varepsilon, (k+1)\varepsilon)$, $k \ge 0$.
Let $E_k = \{k\varepsilon \le f < (k+1)\varepsilon\}$, these are m'ble and disjoint and
$\bigcup_{k \in \mathbb{N}^0} E_k = \{f < \infty\} \subseteq E$, and so $\Gamma(f, E) = \bigcup_{k=0}^{\infty} \Gamma(f, E_k)$. next, $(x, f(x)) \in \Gamma(f, E_k)$ means $(x, f(x)) \in E_k \times [k\varepsilon, (k+1)\varepsilon]$, so $|\Gamma(f, E_k)| \le |E_k \times [k\varepsilon, (k+1)\varepsilon]| = \varepsilon |E_k|$
So $|\Gamma(f, E)| \le \sum_{k=1}^{\infty} |\Gamma(f, E_k)| \le \varepsilon \sum_{k=0}^{\infty} |E_k| = \varepsilon |\{f < \infty\}| \le \varepsilon |E|$, but ε arb.

Case 2: $|E| = \infty$, split into shells $E_k = E \cap (B_{k+1}(0) \setminus B_k(0))$

- Lemma 5.4: Let $E \subseteq \mathbb{R}^n$ m'ble, and $\phi: E \to [0,\infty)$ be a m'ble simple fn. Then
a) $R(\phi, E)$ is m'ble so $\int_E \phi$ exists   b) If $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$, $\int_E \phi = \sum_{j=1}^{N} a_j |E_j|$

Proof: As ϕ m'ble, each E_j m'ble. Let $a_0 = 0$, define $E_0 = E \setminus \bigcup_{j=1}^{N} E_j$ ($\uparrow a_j$ to here). which is m'ble, and $\phi = \sum_{j=0}^{N} a_j \chi_{E_j}$. Then $R(\phi, E) = \{(x,y): x \in E,\ 0 \leq y \leq \phi(x)\}$
$= \bigcup_{j=0}^{N} \{(x,y) : x \in E_j,\ 0 \leq y \leq a_j\} = \bigcup_{j=0}^{N} E_j \times [0, a_j]$. Each $E_j \times [0,a_j]$ is m'ble and disjoint so $R(\phi, E)$ is m'ble, and $\int_E \phi = |R(\phi,E)| = \sum_{j=0}^{N} |E_j \times [0,a_j]| = \sum_{j=1}^{N} a_j |E_j|$ □

Proof of Thm 5.1 (c): Let $f: E \to [0,\infty]$ is m'ble. By thm 4.13, $\exists$ an increasing seq of m'ble simple fns $\{\phi_k\}$ st. $\lim_{k\to\infty} \phi_k = f$ on E. $R(\phi_k, E)$ is m'ble by lemma above.
We claim $R(\phi_k, E) \cup \Gamma(f,E) \nearrow R(f,E)$. As $\phi_k \leq \phi_{k+1}$ and these only take finite values, $R(\phi_k, E) \subseteq R(\phi_{k+1}, E)$. Next, let $x \in R(f,E)$, we will show $R(f,E) \subseteq \bigcup_{k=1}^{\infty} (R(\phi_k,E) \cup \Gamma(f,E))$
Case 1: $f(x) < \infty$. Then $0 \leq y \leq f(x)$. If $y < f(x)$, $\lim_{k\to\infty} \phi_k(x) = f(x) > y$, so for some large enough k, $y < \phi_k(x)$ so $(x,y) \in R(\phi_k, E)$. If $y = f(x)$, $(x,y) \in \Gamma(f,E)$.
Case 2: $f(x) = \infty$, then $0 \leq y \leq f(x)$ and again for large enough k, $y \leq \phi_k$.
Hence $R(\phi_k, E) \cup \Gamma(f,E) \nearrow R(f,E)$ and so $R(f,E)$ is m'ble. □

- Thm 5.5: Let f, g nonnegative, m'ble on E m'ble.
i) If $g \leq f$ on E, $\int_E g \leq \int_E f$ ii) If $\int_E f$ finite, then f finite a.e.
iii) If $E_1 \subseteq E_2 \subseteq E$, E_1, E_2 m'ble then $\int_{E_1} f \leq \int_{E_2} f$.

Proof: i) As $g \leq f$, $R(g,E) \subseteq R(f,E)$. iii) As $E_1 \subseteq E_2$, $R(f,E_1) \subseteq R(f,E_2)$.
ii) Let $A = \{f = \infty\}$. First A m'ble. Let $a > 0$. Then $a|A| = \int a \chi_A \leq \int_A f \leq \int_E f$, so $|A| \leq \frac{1}{a} \int_E f$ $\forall a > 0$, hence $|A| = 0$. □

- Thm 5.6: (Monotone Convergence Thm): Let $\{f_k\}$ m'ble, nonnegative, on m'ble E st. $f_k \nearrow f$ on E. Then f m'ble and $\lim_{k\to\infty} \int_E f_k = \int_E \lim_{k\to\infty} f_k = \int_E f$.

Proof: f is m'ble by thm 4.12, and $f \geq 0$. Similarly to thm 5.1 proof, $R(f_k, E) \cup \Gamma(f,E) \nearrow R(f,E)$ (we did not use the fact that ϕ_k were simple). By thm 3.26, $\lim_{k\to\infty} |R(f_k,E) \cup \Gamma(f,E)| = |R(f,E)|$. Since $|\Gamma(f,E)| = 0$, $|R(f_k,E)| \leq |R(f_k,E) \cup \Gamma(f,E)| \leq |R(f_k,E)| + 0$, so $|R(f_k,E) \cup \Gamma(f,E)| = |R(f_k,E)|$.
Then $\lim_{k\to\infty} \int_E f_k = \lim_{k\to\infty} |R(f_k, E)| = |R(f,E)| = \int_E f$.

- thm 5.7 (Disjoint sum): Suppose $E \subseteq \mathbb{R}^n$ m'ble and $E = \bigcup_{k=1}^{\infty} E_k$ disjoint, m'ble.
  Let $f: E \to [0,\infty]$ m'ble, then $\int_E f = \sum_{k=1}^{\infty} \int_{E_k} f$
  Proof: If $x \in E$, $x \in E_j$ for exactly one E_j. So $R(f,E) = \bigcup_{k=1}^{\infty} R(f,E_k)$ is a disjoint union of m'ble sets. So $|R(f,E)| = \sum_{k=1}^{\infty} |R(f,E_k)|$ □
- thm 5.8: Suppose $E \subseteq \mathbb{R}^n$ m'ble, $f: E \to [0,\infty]$ m'ble. Then
  $\int_E f = \sup \sum_j [\inf_{E_j} f] \, |E_j|$ (*), when sup is taken over all decompositions $E = \bigcup E_j$ into finite # of disjoint m'ble sets
- thm 5.9: Suppose $|E| = 0$. Let $f: E \to [0,\infty]$. Then f is m'ble and $\int_E f = 0$.
  Proof: $\forall a \in \mathbb{R}$, $\{f > a\} \subseteq E$ so $|\{f>a\}| \le 0$, hence f is m'ble.
  Next, $R(f,E) \subseteq E \times [0,\infty) = \bigcup_{k=0}^{\infty} E \times [k,k+1)$. Then $|R(f,E)| \le \sum_{k=0}^{\infty} |E \times [k,k+1)| = \sum_{k=0}^{\infty} |E| \cdot 1 = 0$. □
- thm 5.10: Let $E \subseteq \mathbb{R}^n$ m'ble, $f,g: E \to [0,\infty]$ m'ble.
  a) If $g \le f$ a.e, then $\int_E g \le \int_E f$
  b) If $g = f$ a.e. then $\int_E g = \int_E f$
  Proof: (a) Let $A = \{g \le f\}$, $Z = \{g > f\}$, we know $|Z| = 0$. Since A, Z m'ble & disj,
  $\int_E g = \int_A g + \int_Z g = \int_A g \le \int_A f \le \int_E f$.
  (b) By (a), $\int_E g \le \int_E f$ and $\int_E f \le \int_E g$. □
- thm 5.11: Let $E \subseteq \mathbb{R}^n$ m'ble, $f: E \to [0,\infty]$ m'ble. Then $\int_E f = 0$ iff $f = 0$ a.e.
  Proof: ($\Leftarrow$): By thm 5.10b, $\int_E f = \int_E 0 = 0$.
  ($\Rightarrow$): Let $a > 0$. Then $a|\{f>a\}| = \int_{\{f>a\}} a \chi_{\{f>a\}} \le \int_{\{f>a\}} f \le \int_E f = 0$.
  So $\forall a \in \mathbb{R}^+$, $|\{f>a\}| = 0$. Then $|\{f>0\}| = |\bigcup_{k \in \mathbb{N}} \{f > \frac{1}{k}\}| \le \sum_{k \in \mathbb{N}} 0 = 0$. □
- Corollary 5.12 (Chebyshev Inequality): Let E m'ble, $f: E \to [0,\infty]$ m'ble, let $a > 0$,
  then $|\{f > a\}| \le \frac{1}{a} \int_E f$.
  Proof: $a|\{f>a\}| = \int_{\{f>a\}} a \chi_{\{f>a\}} \le \int_{\{f>a\}} f \le \int_E f$, divide by a. □
- thm 5.13: Let $E \subseteq \mathbb{R}^n$ m'ble, $f: E \to [0,\infty]$ m'ble. Let $c > 0$, then $\int_E cf = c\int_E f$.
  Proof: By thm 4.13, $\exists$ nonnegative simple fns $\phi_k \nearrow f$. Then $c\phi_k \nearrow cf$ and $c\phi_k$ is still nonnegative, simple. By the formula in lemma 5.4,
  $\int_E c\phi_k = c \cdot \int_E \phi_k$. By MCT, $\int_E cf = \lim_{k\to\infty} \int c\phi_k = c \lim_{k\to\infty} \int \phi_k = c\int_E f$. □
- thm 5.14: Let $E \subseteq \mathbb{R}^n$, $f,g: E \to [0,\infty]$ m'ble. Then $\int_E (f+g) = \int_E f + \int_E g$

Proof: Case 1: f, g simple. Write $f = \sum_{i=1}^{N} a_i \chi_{A_i}$, $g = \sum_{j=1}^{M} b_j \chi_{B_j}$. Then note $f+g$ is simple (it takes finitely many values), and $f+g = \sum_{i=1}^{N} \sum_{j=1}^{M} (a_i + b_j) \chi_{A_i \cap B_j}$. Then,
$\int_E (f+g) = \sum_{i=1}^{N} \sum_{j=1}^{M} (a_i + b_j) |A_i \cap B_j| = \sum_{i=1}^{N} a_i \sum_{j=1}^{M} |A_i + B_j| + \sum_{j=1}^{M} b_j + \sum_{i=1}^{N} |A_i + B_j|$
$= \sum_{i=1}^{N} a_i |A_i| + \sum_{j=1}^{M} b_j |B_j| = \int_E f + \int_E g$.

Case 2: general m'ble, nonnegative fns f, g. $\exists$ simple, m'ble, nonneg. fns $\phi_k \nearrow f$, $\psi_k \nearrow g$. Then $\phi_k + \psi_k$ is simple and $(\phi_k + \psi_k) \nearrow (f+g)$. By MCT, $\int_E (f+g) = \lim_{k\to\infty} \int_E (\phi_k + \psi_k)$
$= \lim_{k\to\infty} (\int_E \phi_k + \int_E \psi_k) = \int_E f + \int_E g$. $\square$

- Corollary 5.15: Let $E \subseteq \mathbb{R}^n$ m'ble, $f, g: E \to [0,\infty)$ m'ble, w/ $0 \le f \le g$ & $\int_E g < \infty$. Then $\int_E (g-f) = \int_E g - \int_E f$.

Proof: Note $g - f \ge 0$ and m'ble. Also, $g = (g-f) + f$ so $\int_E g = \int_E f + \int_E g - f$ $\square$

- thm 5.16: let $E \subseteq \mathbb{R}^n$ m'ble, $f_k: E \to [0,\infty]$ m'ble. Then $\int_E \sum_{k} f_k = \sum_{k\in\mathbb{N}} \int_E f_k$.

Proof: let $F_N = \sum_{k=1}^{N} f_k$ $(N \ge 1)$, and $F = \sum_{k=1}^{\infty} f_k$. F_N m'ble (finite sum) and nonnegative. $F_N \nearrow F$, and F is m'ble, and by MCT, $\int_E F = \lim_{N\to\infty} \int_E F_N = \lim_{N\to\infty} \int_E \sum_{k=1}^{N} f_k = \lim_{N\to\infty} \sum_{k=1}^{N} \int_E f_k = \sum_{k=1}^{\infty} \int_E f_k$ $\square$

- thm 5.17 (Fatou's Lemma / The Integral is Lower Semicontinuous): Let $E \subseteq \mathbb{R}^n$ m'ble, $\{f_k\}$ be nonnegative and m'ble. Then $\liminf_{k\to\infty} \int_E f_k \ge \int_E (\liminf_{k\to\infty} f_k)$

Proof: define $g_k = \inf_{j \ge k} f_j$, each g_k is m'ble and $0 \le g_1 \le g_2 \le \dots$ and $g_k \nearrow \liminf f_k$. By Monotone Convergence Thm, $\int_E \liminf f_k = \lim \int_E g_k = \liminf \int_E g_k \le \liminf \int_E f_k$. $\square$

- thm 5.19 (Dominated Convergence Thm): Let $E \subseteq \mathbb{R}^n$ m'ble, $f_k: E \to [0,\infty]$ be m'ble. Assume a) $\lim_{k} f_k = f$ a.e. on E b) There exists a m'ble fn $\phi: E \to [0,\infty]$ s.t. $\int_E \phi < \infty$ and every $f_k \le \phi$ a.e.. Then $\lim_{k\to\infty} \int_E f_k = \int_E \lim f_k = \int_E f$.

Proof: By Fatou's Lemma, $\liminf \int_E f_k \ge \int_E \liminf f_k = \int_E \lim f_k = \int_E f$.
Next, $\{\phi - f_k\}$ is m'ble and nonnegative so by Fatou's Lemma again, $\liminf \int_E (\phi - f_k) \ge \int_E \liminf (\phi - f_k) = \int_E (\phi - f)$. Since $0 \le f_k \le \phi$, $\int_E \phi < \infty$, by Cor. 5.15, $\liminf \int_E \phi - f_k = \liminf (\int_E \phi - \int_E f_k) = \int_E \phi - \limsup \int_E f_k$.
Likewise, $\int_E \phi - f = \int_E \phi - \int_E f$. Substituting, $\int_E f \ge \limsup \int_E f_k$, so $\int_E f \le \liminf \int_E f_k \le \limsup \int_E f_k \le \int_E f$, hence $\int_E f = \lim \int_E f_k$. $\square$

- def: Let $E \subseteq \mathbb{R}^n$ m'ble, $f: E \to \overline{\mathbb{R}}$ m'ble. Write $f = f^+ - f^-$.
  a) If at least one of $\int_E f^+$, $\int_E f^-$ is finite, define $\int_E f = \int_E f^+ - \int_E f^-$, we say $\int_E f$ exists.
  b) If $\int_E f$ exists and is finite, f is (Lebesgue) integrable on E, we write $f \in L_1(E)$

- thm 5.20, 5.21: Let $E \subseteq \mathbb{R}^n$ be m'ble, $f: E \to \overline{\mathbb{R}}$ m'ble.
  a) If $\int_E f$ exists, then $|\int_E f| \le \int_E |f|$ b) $f \in L_1(E)$ iff $|f| \in L_1(E)$ iff $f^+, f^- \in L_1(E)$
  Proof: (a): $|\int_E f| = |\int_E f^+ - f^-| \le |\int_E f^+| + |\int_E f^-| = \int_E f^+ + \int_E f^- = \int_E f^+ + f^- = \int_E |f|$
  (b): $\int_E f$ finite $\Leftrightarrow$ $\int_E f^+$, $\int_E f^-$ finite $\Leftrightarrow$ $\int_E f^+ + f^- = \int_E |f|$ finite □
- thm 5.22: let $E \subseteq \mathbb{R}^n$ m'ble, $f: E \to \overline{\mathbb{R}}$ be integrable. Then f is finite a.e..
  Proof: By thm 5.21, $\int_E |f|$ finite. So $|f|$ finite a.e and so f finite a.e. (thm 5.5) □
- thm 5.23: let $E \subseteq \mathbb{R}^n$ m'ble
  a) If $\int_E f$, $\int_E g$ exists and $f \le g$ a.e., then $\int_E f \le \int_E g$
  b) If $A \subseteq E$ m'ble and $\int_E f$ exists, then $\int_A f$ exists
  c) If $A \subseteq E$ m'ble and $\int_E f$ finite, then $\int_A f$ finite
  Proof: (a): As $f \le g$ a.e. $f^+ = \max\{0, f\} \le \max\{0, g\} = g^+$. a.e
  $f^- = -\min\{0, f\} \ge -\min\{0, g\} = g^-$ a.e. By thm 5.10, $\int_E f^+ \le \int_E g^+$, $\int_E f^- \ge \int_E g^-$,
  then, $\int_E f = \int_E f^+ - \int_E f^- \le \int_E g^+ - \int_E g^- = \int_E g$.
  (b): Since $\int_E f$ exists, at least one of $\int_E f^+$, $\int_E f^-$ is finite. Thus
  at least one of $\int_A f^+$, $\int_A f^-$ is finite. Hence $\int_A f$ exists.
  (c): Since $\int_E f$ finite, then $\int_E f^+$, $\int_E f^-$ are finite. So $\int_A f^+$, $\int_A f^-$
  are finite and $\int_A f$ is finite. □
- Corollary: If $\int_E f$, $\int_E g$ exist and $f = g$ a.e., then $\int_E f = \int_E g$
- thm 5.24: Let $E \subseteq \mathbb{R}^n$ m'ble. Write $E = \bigcup_{k=1}^{\infty} E_k$, disjoint, m'ble. If $f: E \to \mathbb{R}$ is integrable
  then $\int_E f = \sum_{k=1}^{\infty} \int_{E_k} f$.
  Proof: As $\int_E f$ finite, $\int_{E_k} f$ finite. So $\int_{E_k} f^+$, $\int_{E_k} f^-$ finite, then $\int_E f^+ = \sum_{k=1}^{\infty} \int_{E_k} f^+$
  and $\int_E f^- = \sum_{k=1}^{\infty} \int_{E_k} f^-$. As f is integrable, $\int_E f^+$, $\int_E f^-$ are finite, so the series converge.
  Thus $\int_E f = \int_E f^+ - \int_E f^- = \left(\sum_{k=1}^{\infty} \int_{E_k} f^+\right) - \left(\sum_{k=1}^{\infty} \int_{E_k} f^-\right) = \sum_{k=1}^{\infty} \left(\int_{E_k} f^+ - \int_{E_k} f^-\right) = \sum_{k=1}^{\infty} \int_{E_k} f$ □
- Note: this still holds if $\int_E f$ exists (not necessarily finite)
- thm 5.25: If $f = 0$ a.e. in m'ble E then $\int_E f = 0$.
  Proof: Then $|f| = 0$ a.e. so $f^+, f^- = 0$ a.e. hence $\int_E f = \int_E f^+ - \int_E f^- = 0$. □
- Lemma 5.26: Let $\int_E f$ exists. Then $\int_E (-f) = -\int_E f$.
  Proof: $(-f)^+ = \max(0, -f) = -\min(0, f) = f^-$, $(-f)^- = -\min(0, -f) = \max(0, f) = f^+$
  At least one of $f^+ = (-f)^-$ and $f^- = (-f)^+$ is finite, so $\int_E f$ exists and
  $\int_E (-f) = \int_E (-f)^+ - \int_E (-f)^- = \int_E f^- - \int_E f^+ = -(\int_E f^+ - \int_E f^-) = -\int_E f$ □

- Thm 5.27: Let $\int_E f$ exist and $c \in \mathbb{R}$. Then $\int_E cf$ exists and $\int_E cf = c\int_E f$.

Proof: Case 1: $c \geq 0$. Then $(cf)^+ = cf^+$, $(cf)^- = cf^-$. At least one of $\int_E cf^+ = c\int_E f^+$ and $\int_E cf^- = c\int_E f^-$ is finite, and $\int_E cf = c(\int_E f^+ - \int_E f^-) = c\int_E f$.
Case 2: $c < 0$, then write $cf = (-1)|c|f$ and use the previous case/lemma. □

- Thm 5.28: Let $f, g \in L_1(E)$. Then $\int f+g$ is integrable and $\int_E (f+g) = \int_E f + \int_E g$

Proof: We know $|f|, |g| \in L_1(E)$ as well, so $|f|+|g| \in L_1(E)$. Next, $f+g$ and $|f+g|$ is m'ble. Also $|f+g| \geq 0$ so $\int_E |f+g|$ exists. Then $\int_E |f+g| \leq \int_E |f| + |g| = \int_E |f| + \int_E |g| < \infty$, so $|f+g|$ is integrable and $f+g$ is integrable as well, see notes for rest. □

- Corollary: We have linearity for integrals for a finite sum
- Corollary 5.29: Let f m'ble and ϕ integrable on E. Assume $f \geq \phi$ a.e. Then $\int_E (f-\phi) = \int_E f - \int_E \phi$. (Note: $\int_E f = \infty$ is possible)

Proof: Case 1: f is integrable, so $\int_E f$ finite. Then $\int_E f - \phi = \int_E (f + (-1)\phi) = \int_E f - \int_E \phi$.
Case 2: f is not integrable, so $\int_E f$ is not finite. Now $f^- = -\min(0,f) \leq -\min(0,\phi) = \phi^-$. So $\int_E f^- \leq \int_E \phi^- < \infty$, meaning $\int_E f^+ = \int_E f$ must be ∞. Note, $f - \phi \geq 0$, so $\int f - \phi$ exists. We claim the integral is ∞. If it was finite, $\int_E f = \int_E f - \phi + \phi = \int_E (f-\phi) + \int \phi$ which is finite. Then $\int_E (f-\phi) = \infty = \infty - \int_E \phi = \int_E f - \int_E \phi$. □

- Thm 5.30: Let f integrable on E, g m'ble, and for some $M > 0$, $|g| \leq M$ a.e. on E. Then fg is integrable.

Proof: First, fg, $|fg|$ are m'ble. Also, $|fg| \leq M|f|$ a.e. in E, so $\int_E |fg| \leq \int_E M|f| = M\int_E |f| < \infty$, so $|fg|$ is integrable, hence fg is integrable. □

- Corollary 5.31: Let f integrable, $f \geq 0$ a.e. in E. Let g m'ble and for some $\alpha, \beta \in \mathbb{R}$, $\alpha \leq g \leq \beta$ a.e. in E. Then $\alpha\int_E f \leq \int_E fg \leq \beta\int_E f$.
- Thm 5.32 (General Monotone Convergence Thm): Let $\{f_k\}$ be m'ble fns on E.
  a) Assume $f_k \nearrow f$ a.e. and $\exists\, \phi$ integrable s.t. $f_k \geq \phi$ a.e. for all k. Then $\lim_{k\to\infty} \int_E f_k = \int_E f$.
  b) Assume $f_k \searrow f$ a.e. and $\exists\, \phi$ integrable s.t. $f_k \leq \phi$ a.e. for all k. Then $\lim_{k\to\infty} \int_E f_k = \int_E f$.

Proof: (a) We may assume $f_k \nearrow f$ and $f_k \geq \phi$ everywhere (remove the $|\cdot| = 0$ set where this is not true, which doesn't change the integral). Then $f_k - \phi \geq 0$ and $f_k - \phi \nearrow f - \phi$, and by the Monotone Convergence Thm, $\lim_{k\to\infty} \int_E (f_k - \phi) = \int_E (f-\phi)$. Since ϕ is integrable, by Cor 5.29,
$\lim_{k\to\infty} \int_E f_k - \underbrace{\int_E \phi}_{<\infty} = \int_E f - \underbrace{\int_E \phi}_{<\infty}$, so $\lim_{k\to\infty} \int_E f_k = \int_E f$

(b) Apply (a) with $\{-f_k\}$. □

- thm 5.33 (Uniform Convergence thm): Assume $|E|<\infty$. Let $\{f_k\}$ integrable and converge uniformly to f on E, i.e. $\sup_E |f_k - f| \to 0$, as $k\to\infty$. Then f integrable and $\lim \int_E f_k = \int_E f$.

Proof: By thm 4.12, f is m'ble. By uniform convergence, for large enough k, $|f| = |(f-f_k)+f_k| \le |f-f_k| + |f_k| \le 1 + |f_k|$. Then $\int_E |f| \le \int_E 1 + |f_k| = |E| + \int_E |f_k| < \infty$, so $|f|$ is integrable and so is f.
By thm 5.28, $|\int_E f - \int_E f_k| = |\int_E (f-f_k)| \le \int_E |f-f_k| \le \int_E \sup_E |f-f_k| = |E| \cdot \sup_E |f-f_k| \xrightarrow{(k\to\infty)} 0$. □

Remark: The proof shows $\lim_{k\to\infty} \int_E |f-f_k| = 0$. (convergence in L^1)

- thm 5.34 (General Fatou's Lemma): Let $\{f_k\}$ m'ble on E, assume $\exists\, \phi$ integrable so that $f_k \ge \phi$ a.e. on E. Then $\liminf_{k\to\infty} \int_E f_k \ge \int_E \liminf_{k\to\infty} f_k$.

Proof: Since $f_k - \phi \ge 0$ everywhere (we may assume), Fatou's lemma gives $\liminf_{k\to\infty} \int_E (f_k - \phi) \ge \int_E \liminf_{k\to\infty} (f_k - \phi)$. As $\int_E \phi$ is finite, Cor 5.29 gives $\liminf (\int_E f_k - \int_E \phi) \ge \int_E ((\liminf f_k) - \phi) = \int_E \liminf f_k - \int_E \phi$, so $\liminf \int_E f_k \ge \int_E \liminf f_k$ □

Remark: If $f \le \phi$ a.e., ϕ integrable, then $\limsup \int_E f_k \le \int_E \limsup f_k$ (apply 5.34 to $\{-f_k\}$)

- thm 5.36 (General Dominated Convergence Thm): Let $E \subseteq \mathbb{R}^n$, $f_k : E \to \overline{\mathbb{R}}$ m'ble. Assume $\lim_{k\to\infty} f_k = f$ a.e. and $\exists\, \phi$ integrable s.t. $|f_k| \le \phi$ a.e. in E. Then $\lim_{k\to\infty} \int_E f_k = \int_E f$.

Proof: Now $-\phi \le f_k \le \phi$ a.e. in E, so $0 \le f_k + \phi \le 2\phi$ a.e. in E.
Since 2ϕ is integrable and $f_k + \phi \ge 0$ and $\lim (f_k + \phi) = f + \phi$ in E, DCT gives $\lim_{k\to\infty} \int_E (f_k + \phi) = \int_E (f + \phi)$. Note $|f_k| \le \phi$ a.e. $\Rightarrow |f| \le \phi$ a.e. so f is integrable. Then $\lim (\int_E f_k + \int_E \phi) = \int_E f + \int_E \phi \Rightarrow \lim \int_E f_k = \int_E f$. □

- Cor 5.37: Let $\{f_k\}$ m'ble on E, and $|E|<\infty$. Assume $\lim_{k\to\infty} f_k = f$ a.e. in E and $\exists M>0$ s.t. $|f_k| \le M$ a.e. in E. Then $\lim_{k\to\infty} \int_E f_k = \int_E f$

Proof: Apply thm 5.36 with $\phi = M$. □

- def: Let $E \subseteq \mathbb{R}^n$ m'ble, $f : E \to \mathbb{R}$ m'ble. The distribution fn of f on E is $\omega(\alpha) = |\{f > \alpha\}|$. For this section, we'll assume $|E|<\infty$ and f finite a.e.

Properties: a) $0 \le \omega(\alpha) \le |E|$ b) ω is decreasing in α c) $\lim_{\alpha\to\infty} \omega(\alpha) = 0$
d) $\lim_{\alpha\to-\infty} \omega(\alpha) = |E|$ e) $\omega \in BV[\mathbb{R}]$, $V[\omega; -\infty, \infty] = |E|$.

Proof: b) If $\alpha \le \beta$, then $\{f>\alpha\} \supseteq \{f>\beta\}$, so $\omega(\alpha) = |\{f>\alpha\}| \ge |\{f>\beta\}| = \omega(\beta)$
c) Note $\{f>k\} \searrow \{f=\infty\}$ so $\omega(k) \searrow |\{f=\infty\}| = 0$ hence $\lim_{k\to\infty} \omega(k) = 0$. Since ω is decreasing, $\lim_{\alpha\to\infty} \omega(\alpha) = 0$. d) Same as c

e) As ω is decreasing on any interval $[b,c]$, $V[\omega; b,c] = \omega(b) - \omega(c) \le |E| - 0$.
Taking $b \to -\infty$, $c \to \infty$, $V[\omega; -\infty, \infty] \le |E|$. $\square$

- Lemma 5.38: If $\alpha < \beta$, then $\omega(\alpha) - \omega(\beta) = |\{\alpha < f \le \beta\}|$

Proof: $\omega(\alpha) - \omega(\beta) = |\{f > \alpha\}| - |\{f > \beta\}| = |\{f > \alpha\} \setminus \{f > \beta\}| = |\{\alpha < f \le \beta\}|$ $\square$

- Note: the left/right limits for ω exist as ω is decreasing: $\omega(\alpha-) = \lim_{\varepsilon \to 0^+} \omega(\alpha - \varepsilon)$; $\omega(\alpha+) = \lim_{\varepsilon \to 0^+} \omega(\alpha + \varepsilon)$
- Lemma 5.39: a) $\omega(\alpha+) = \omega(\alpha)$ (ω is right continuous) b) $\omega(\alpha-) = |\{f \ge \alpha\}|$

Proof: a) let $\varepsilon_k \searrow 0$. Then $\{f > \alpha + \varepsilon_k\} \nearrow \{f > \alpha\}$, so $\omega(\alpha+) = \lim_{k\to\infty} |\{f > \alpha + \varepsilon_k\}|$
$= |\{f > \alpha\}| = \omega(\alpha)$.

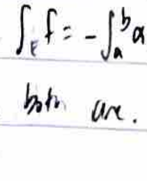
b) let $\varepsilon_k \searrow 0$. Then $\{f > \alpha - \varepsilon\} \searrow \{f \ge \alpha\}$, so $\omega(\alpha-) = \lim_{k\to\infty} |\{f > \alpha - \varepsilon\}| = |\{f \ge \alpha\}|$. $\square$

- Corollary 5.40: a) $\omega(\alpha-) - \omega(\alpha+) = |\{f = \alpha\}|$. In particular, ω is cont. @ α iff $|\{f = \alpha\}| = 0$. b) ω is constant on (α, β) iff $|\{\alpha < f < \beta\}| = 0$.

Proof: a) Note $\{f > \alpha\}$ and $\{f = \alpha\}$ are disjoint and m'ble, and so
$|\{f \ge \alpha\}| = |\{f > \alpha\}| + |\{f = \alpha\}|$. By Lemma 5.39, $\omega(\alpha-) = \omega(\alpha+) + |\{f = \alpha\}|$
b) $|\{\alpha < f < \beta\}| = |\{f > \alpha\}| - |\{f \ge \beta\}| = \omega(\alpha) - \omega(\beta-) = 0$ iff ω is constant on (α, β)

- Theorem 5.41: (Lebesgue Integrals & Distribution Functions). Assume $|E| < \infty$ and f m'ble. Let $-\infty < a < b < \infty$ and $a < f(x) < b$ for $x \in E$. Then $\int_E f = -\int_a^b \alpha \, d\omega(\alpha)$

Pf: Take a partition $a = \alpha_0 < \alpha_1 < \dots < \alpha_k = b$ of $[a,b]$. Let $E_j = \{\alpha_{j-1} < f \le \alpha_j\}$.
Then E_j is m'ble and by thm 5.23, $\alpha_{j-1} |E_j| \le \int_{E_j} f \le \alpha_j |E_j|$ (*)



By lemma 5.38, $|E_j| = \omega(\alpha_{j-1}) - \omega(\alpha_j)$. Also note the E_j are distinct, so $\int_E f = \sum_{j=1}^{k} \int_{E_j} f$. Then taking the sum over all j in (*),
$\sum_{j=1}^{k} (-\alpha_{j-1}(\omega(\alpha_j) - \omega(\alpha_{j-1}))) \le \int_E f \le \sum_{j=1}^{k} -(\alpha_j(\omega(\alpha_j) - \omega(\alpha_{j-1})))$. Then note as $\max_{1 \le j \le k} (\alpha_j - \alpha_{j-1}) \to 0$, the left and right side $\to -\int_a^b \alpha \, d\omega(\alpha)$, and so $\int_E f = -\int_a^b \alpha \, d\omega(\alpha)$. $\square$

- Theorem 5.43: If either $\int_E f$ or $\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha)$ exists and is finite, then both are. Moreover, $\int_E f = -\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha)$.

- Theorem 5.46: Let $-\infty < a < b < \infty$, f m'ble in m'ble E, $a < f \le b$ in E, let $\phi : [a,b] \to \mathbb{R}$ cont. Assume $|E| < \infty$, then $\int_E \phi \circ f = -\int_a^b \phi(\alpha) \, d\omega(\alpha)$.

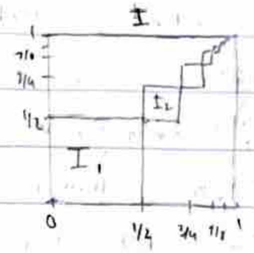
- Theorem 5.47: (General Distribution Formula): Let $\phi : \mathbb{R} \to \mathbb{R}$ cont, f m'ble, $\phi(f)$ integrable on E. If $|E| < \infty$, then $\int_{-\infty}^{\infty} \phi(\alpha) \, d\omega(\alpha)$ exists and $\int_E \phi \circ f = -\int_{-\infty}^{\infty} \phi(\alpha) \, d\omega(\alpha)$

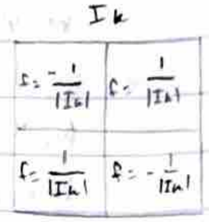
- Lemma 5.49 (Chebyshev in L^p): Let $0 < p < \infty$, $f \in L^p(E)$. Then for $\alpha > 0$,

$$\omega(\alpha) = |\{f > \alpha\}| \le \frac{1}{\alpha^p} \int_{\{f > \alpha\}} |f|^p$$

- **theorem 5.51** Let $0 < p < \infty$, $f \geq 0$, $\int_E f^p < \infty$. Then $\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) = p\int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha$
- **Notation**: Let I_1, I_2 be closed intervals in $\mathbb{R}^n, \mathbb{R}^m$ respectively. Let $I = I_1 \times I_2$, then a function $f: I \to \mathbb{R}$ is represented $f(\vec{x}, \vec{y})$ and the integral $\iint_I f(\vec{x},\vec{y}) \, d\vec{x} \, d\vec{y}$. For $\vec{x} \in I_1$, define the slice function $f_{\vec{x}}(\vec{y}) = f(\vec{x},\vec{y})$ for $\vec{y} \in I_2$ (Similarly for $\vec{y}$)
- **theorem 6.1** (Fubini's Theorem): Let $f(\vec{x},\vec{y})$ be integrable over $I = I_1 \times I_2$, so $\iint_I |f| < \infty$. Then i) for a.e. $x \in I_1$, $f_{\vec{x}}$ is m'ble, integrable in I_2.
ii) the function $F(x) = \int_{I_2} f(\vec{x},\vec{y}) \, dy$ (for $\vec{x} \in I_1$) is m'ble and integrable in I_1
iii) $\iint_I f(x,y) \, dx dy = \int_{I_1} \int_{I_2} f(x,y) \, dy dx$
Remarks: a) If f is integrable over I, then double integral = iterated/repeated integral
b) We can set $f = 0$ outside I, still m'ble and can work on $\mathbb{R}^n \times \mathbb{R}^m$ instead
- **def**: f has Property $\mathcal{F}$ if it satisfies the conclusion of Fubini's theorem
- **Lemma 6.2**: A finite linear combinations of functions with property $\mathcal{F}$ has prop $\mathcal{F}$
- **lemma 6.3**: Let $\{f_k\}$ have prop $\mathcal{F}$. If $f_k \nearrow f$ or $f_k \searrow f$, and f integrable, then f has prop $\mathcal{F}$
- **lemma 6.4**: Let E be a G_δ and $|E| < \infty$. Then χ_E has property $\mathcal{F}$.
- **lemma 6.5**: If $Z \subseteq \mathbb{R}^{n+m}$ and $|Z| = 0$, then χ_Z has property $\mathcal{F}$.
- **lemma 6.6**: Let $E \subseteq \mathbb{R}^{n+m}$ be measurable, $|E| < \infty$. Then χ_E has property $\mathcal{F}$.
Proof: $E = H \setminus Z$ for H G_δ and $|Z| = 0$. WMA $Z \subseteq H$ ($Z = Z \cap H$), so $|E| = |H| - |Z| = |H|$. χ_H, χ_Z have prop $\mathcal{F}$ by previous lemmas then $\chi_E = \chi_H - \chi_Z$ has property $\mathcal{F}$. □
- **Proof of Fubini's thm**: Let $f: \mathbb{R}^{n+m} \to \mathbb{R}$ integrable. Write $f = f^+ - f^-$. Note f^+, f^- are nonnegative and integrable. Let $\{\phi_k\}$ be nonnegative simple fns such that $\phi_k \nearrow f^+$. Since $0 \leq \phi_k \leq f^+$ and f^+ integrable, ϕ_k are integrable. Note $\phi_k = \sum_{j=1}^{l} c_j \chi_{E_j}$ for m'ble disjoint E_j. By Lemma 5.4, we have $\infty > \iint_E \phi_k = \sum_{j=1}^{l} c_j |E_j|$ and so all $|E_j| < \infty$ ($c_j > 0$). By Lemma 6.6, χ_{E_j} has property $\mathcal{F}$, so by lemma 6.2, ϕ_k has property $\mathcal{F}$, then by lemma 6.3, as $\phi_k \nearrow f^+$, f^+ has property $\mathcal{F}$ as desired. Likewise for f^-, then $f = f^+ - f^-$ has property $\mathcal{F}$. □
- **theorem 6.7 6.8** (General Fubini's Thm): Let $E \subseteq \mathbb{R}^{n+m}$, $f: E \to \overline{\mathbb{R}}$ m'ble. For each $x \in \mathbb{R}^n$, let $E_x = \{y \in \mathbb{R}^m : (x,y) \in E\}$. a) for $x \in \mathbb{R}^n$ a.e., $f_x: E_x \to \overline{\mathbb{R}}$ m'ble. b) If f integrable on E, then i) for $x \in \mathbb{R}^n$ a.e., f_x integrable over E_x ii) $F(x) = \int_{E_x} f(x,y) \, dy$ exists a.e., m'ble, integrable. iii) $\iint_E f(x,y) \, dx dy = \int_{\mathbb{R}^n} \left[\int_{E_x} f(x,y) \, dy \right] dx$.

note: If $f(x,y)$ is integrable over I, then the repeated/iterated integral are finite. We now show the converse is not true.





ex: Let $I = [0,1] \times [0,1]$; choose I_k's as pictured, so sides of I_k have length $\frac{1}{2^k}$ and area $\frac{1}{2^{2k}}$. Note the sides of the I_k fill $[0,1)$. Divide each I_k into 4 squares and define f on the interior as pictured. Let $f=0$ everywhere else. We claim for each $x \in [0,1]$ that $\int_0^1 |f(x,y)|\,dy < \infty$ and $\int_0^1 f(x,y)\,dy = 0$.

Fix $x \in [0,1]$. First suppose there does not exist y s.t. $(x,y) \in \operatorname{int}(I_k)$ for some k. Then $f(x,y) = 0$, so both hold. Now suppose there is some y and k. Then we see this can only happen for one k. We can write $I_k = [a_k, b_k] \times [a_k, b_k]$ and $b_k - a_k = \frac{1}{2^k}$. Next, $|f(x,y)| = \frac{1}{|I_k|}$ for $y \in (a_k, b_k)$ except at the midpoint, so $\int_0^1 |f(x,y)|\,dy = \int_{a_k}^{b_k} \frac{1}{|I_k|}\,dy = 2^k < \infty$. But $\int_0^1 f(x,y)\,dy = 0$ by construction. And so $\int_0^1\int_0^1 f(x,y)\,dy\,dx = 0 = \int_0^1\int_0^1 f(x,y)\,dx\,dy$

however $\int_I f = \sum_{k=1}^{\infty} \int_{I_k} |f| + \int_{I - \bigcup_{k=1}^{\infty} I_k} |f| = \sum_{k=1}^{\infty} (|I_k| \cdot \frac{1}{|I_k|}) + 0 = \infty$. Similarly $\int_I f^+, \int_I f^- = \infty$.

$\ast$ $\int_I f$ does not exist, but the repeated integrals do and are finite (0)

- theorem 6.10 (Tonelli's Thm): Let $E = A \times B$, A, B m'ble in $\mathbb{R}^n, \mathbb{R}^m$ respectively. Let $f: E \to [0,\infty]$ m'ble. Then i) for a.e. $x \in A$, $f_x: B \to [0,\infty]$ is m'ble (not nec. integrable) ii) $F(x) = \int_B f(x,y)\,dy$ is m'ble (not nec. integrable) iii) $\int_E f = \int_A \left(\int_B f(x,y)\,dy\right)dx$

Remark: we can have that $\int_E f = \infty$

Proof: i) for $k \geq 1$, define $f_k(x,y) = \begin{cases} \min\{f(x,y), k\} & \text{if } |(x,y)| \leq k \\ 0 & \text{else} \end{cases}$. Note $f_k \geq 0$ m'ble, bounded by k, and vanishes outside a bounded set, so f_k integrable. Also $f_k \nearrow f$ on E. By Fubini on f_k, for a.e. $\vec{x} \in A$, $f_{k\vec{x}}$ is m'ble. Let $Z = \bigcup_{k=1}^{\infty} \{x : (f_k)_x \text{ not m'ble}\}$, then $|Z| = 0$. Since $f_k \nearrow f$, then $(f_k)_x \nearrow f_x$. So f_x m'ble for $x \notin Z$ hence for a.e. $x \in A$.

ii) By Fubini, $F_k(x) = \int_B f_k(x,y)\,dy$ is m'ble, integrable. By MCT, $F_k(x) \nearrow \int_B f(x,y)\,dy = F(x)$, so $F(x)$ m'ble

iii) By MCT, $\int_E f = \lim_{k\to\infty} \int_E f_k = \lim_{k\to\infty} \int_A \left(\int_B f_k(x,y)\,dy\right)dx$ by Fubini

$= \lim_{k\to\infty} \int_A F_k(x)\,dx = \int_A F(x)\,dx = \int_A \left(\int_B f(x,y)\,dy\right)dx$ $\square$

- thm: Let f m'ble on $E = A \times B$, A, B m'ble in $\mathbb{R}^n, \mathbb{R}^m$. TFAE:

a) $\int_E |f| < \infty$ b) $\int_A\int_B |f(x,y)|\,dy\,dx < \infty$ c) $\int_B \int_A |f(x,y)|\,dx\,dy < \infty$

Moreover, then $\int_E |f| = \int_A \int_B |f(x,y)|\,dy\,dx = \int_B \int_A |f(x,y)|\,dx\,dy$

· def: Let $A \subseteq \mathbb{R}^n$ be m'ble and $f: A \to \overline{\mathbb{R}}$ integrable. define $F(E) = \int_E f$ for m'ble $E \subseteq A$. We call this the indefinite integral of f over A.

note: F's domain is the σ-alg of m'ble subsets of A. Since f integrable, $F(E)$ is finite for all $E \subseteq A$ m'ble. Also if $E = \bigcup_{j=1}^{\infty} E_j$ disjoint m'ble, $F(E) = \sum_{j=1}^{\infty} F(E_j)$ (countable additivity)

· def: F is absolutely continuous (wrt Lebesgue measure) if given $\varepsilon > 0$, $\exists \delta > 0$ st. $|E| < \delta \implies |F(E)| < \varepsilon$ (as $|E| \to 0$, $|F(E)| \to 0$)

· thm 7.1 (Integrable fns have absolutely continuous integrals): Let $f: A \to \overline{\mathbb{R}}$ be integrable. The $F(E)$ abs. cont ($E \subseteq A$ m'ble)

Proof: Since $f = f^+ - f^-$ and f^+, f^- integrable, while abs. cont. is preserved by differences, we may assume f is nonnegative. Since f integrable over A, it is finite a.e. (thm 5.5), Next $f\chi_{\{f > k\}} \downarrow f\chi_{\{f = \infty\}} = 0$ a.e., since f integrable, by MCT, $\lim_{k\to\infty} \int_A f\chi_{\{f > k\}} = \int_A 0 = 0$. Let $\varepsilon > 0$. Choose k large enough so $\int_A f\chi_{\{f > k\}} < \frac{\varepsilon}{2}$. Now, for any m'ble $E \subseteq A$, $0 \le \int_E f\chi_{\{f \le k\}} \le \int_E k = k|E|$. Choose $\delta = \frac{\varepsilon}{2k}$, whenever $|E| < \delta$, then $F(E) = \int_E f = \int_E (f\chi_{\{f \le k\}} + f\chi_{\{f > k\}})$
$\le \int_A f\chi_{\{f > k\}} + k|E| < \frac{\varepsilon}{2} + k\delta = \varepsilon$. $\square$

** · thm 7.2: (Lebesgue's Differentiation Thm): Let $f: \mathbb{R}^n \to \mathbb{R}$ integrable. Let Q be a cube (interval (not closed) w/ equal side lengths), center x. Then for a.e. $x \in \mathbb{R}$, $\lim_{Q \downarrow x} F(Q)/|Q| = \lim_{Q \downarrow x} \int_Q f/|Q| = f(x)$. [indef. int. is diff. w/ deriv. $f(x)$ for a.e. $x \in \mathbb{R}^n$]

· Lemma 7.3: Let $f: \mathbb{R}^n \to \mathbb{R}$ integrable. Then $\exists \{C_k\}$ cont., compactly supported (=0 outside of a compact set) fns. s.t. $\lim_{k\to\infty} \int_{\mathbb{R}^n} |f - C_k| = 0$

* · Lemma 7.4 (Simple Vitali Lemma): Let $E \subseteq \mathbb{R}^n$, $|E|_e < \infty$. Let $K = \{Q\}$ be a collection of axis-parallel cubes covering E w/ all $|Q| > 0$. Let $\beta = \frac{1}{2(5^n)}$, then we can find finitely many disjoint cubes $\{Q_k\}^N$ st. $\sum_{j=1}^{N} |Q_j| \ge \beta |E|_e$.

· Note: The $\{Q_j\}^N$ do not necessarily cover E. β only depends on dimension, not E/K.

· def: Let $f: \mathbb{R}^n \to \mathbb{R}$ m'ble, integrable over every bounded cube Q. Define (for $x \in \mathbb{R}^n$), $f^*(x) = \sup \left(\frac{1}{|Q|} \int_Q |f| \right)$ where sup is taken over all axis parallel cubes centered at x. f^* is the Hardy-Littlewood maximal fn.

· Prop 7.6: Let $f, g: \mathbb{R}^n \to \mathbb{R}$ m'ble, integrable over every bounded cube
i) $0 \le f^* \le \infty$ ii) $(f+g)^*(x) \le f^*(x) + g^*(x)$ iii) $\forall c \in \mathbb{R}$, $(cf)^* = |c| f^*$
iv) f^* is lsc (hence m'ble) v) If for some cube Q, $\int_Q |f| > 0$, $\exists c > 0$ st. for $|x|$ sufficiently large, $f^*(x) \ge \frac{c}{|x|^n}$ vi) If f has compact support, $\exists c > 0$ st. for $|x|$ suff. large, $f^*(x) \le \frac{c}{|x|^n}$

- Remark: You can show $\int_{\{x:|x|\geq 1\}} |x|^{-n} = \infty$, and so if (v) applies, $\int_{\mathbb{R}^n} f^* = \infty$. So f^* is integrable unless $f = 0$ ae.

** - Lemma 7.9 (H-L Inequality): let $f: \mathbb{R}^n \to \mathbb{R}$ integrable, $C = 2\cdot 5^n$. For all $\alpha > 0$, $|\{f^* > \alpha\}| \leq \frac{C}{\alpha}\int_{\mathbb{R}^n}|f|$

Proof: Case 1: f has compact support. By Prop 7.6 (vi), as $|x| \to \infty$, $f^*(x) \to 0$. Let $\alpha > 0$, then $E = \{f^* > \alpha\}$ is bounded (as f has compact support) thus has finite measure. For each $x \in E$, $f^*(x) > \alpha$, and so by def $\exists\, Q_x$ cube s.t. $\frac{1}{|Q_x|}\int_{Q_x}|f| > \alpha \Rightarrow |Q_x| < \frac{1}{\alpha}\int_{Q_x}|f|$.
Let $K = \{Q_x : x \in E\}$, this covers E. By Simple Vitali Lemma, we have a finite collection of cubes $\{Q_{x_j}\}_{j=1}^{N}$ disjoint and $\sum_{j=1}^{N}|Q_{x_j}| \geq \beta|E|$ $(\beta = \frac{1}{2\cdot 5^n})$. Then, $|\{f^* > \alpha\}| = |E|$
$\leq \frac{1}{\beta}\sum_{j=1}^{N}|Q_{x_j}| < \frac{1}{\alpha\beta}\sum_{j=1}^{N}\int_{Q_{x_j}}|f| \underset{\text{(disjoint)}}{\leq} \frac{1}{\alpha\beta}\int_{\mathbb{R}^n}|f| = \frac{C}{\alpha}\int_{\mathbb{R}^n}|f|$.
Case 2: f is any integrable fn. We may assume $f \geq 0$ (<0 doesn't change f^*). For $k \geq 1$, let $f_k = f\cdot\chi_{\{x:|x|\leq k\}}$, so each f_k has compact support. Also $f_k \nearrow f$, so $\{f_k^* > \alpha\} \nearrow \{f^* > \alpha\}$.
By continuity of measure, $|\{f^* > \alpha\}| = \lim_{k\to\infty}|\{f_k^* > \alpha\}| \underset{\text{(Case 1)}}{\leq} \limsup_{k\to\infty}\frac{C}{\alpha}\int_{\mathbb{R}^n}|f_k| \leq \frac{C}{\alpha}\int_{\mathbb{R}^n}|f|$ □

Proof of thm 7.2: By lemma 7.3, $\exists\,\{C_k\}$ cont fns. w/ compact support s.t. $\lim_{k\to\infty}\int_{\mathbb{R}^n}|f - C_k| = 0$.
Let $F(Q)$, $F_k(Q)$ be the indefinite integrals of f, C_k. Let $x \in \mathbb{R}^n$ and Q be a cube center @ x arbitrary.
We have $\left|\frac{F(Q)}{|Q|} - f(x)\right| \leq \left|\frac{F(Q)}{|Q|} - \frac{F_k(Q)}{|Q|}\right| + \left|\frac{F_k(Q)}{|Q|} - C_k(x)\right| + |C_k(x) - f(x)|$
For the first term, $\left|\frac{F(Q)}{|Q|} - \frac{F_k(Q)}{|Q|}\right| = \left|\frac{1}{|Q|}\int_Q[f(y) - C_k(y)]dy\right| \leq \frac{1}{|Q|}\int_Q|f(y) - C_k(y)|dy \underset{\text{HL max fn. for } f-C_k}{\leq} (f - C_k)^*(x)$
For the second, $\left|\frac{F_k(Q)}{|Q|} - C_k(x)\right| = \left|\frac{1}{|Q|}\int_Q[C_k(y) - C_k(x)]dy\right| \leq \frac{1}{|Q|}\cdot|Q|\sup_{y\in Q}|C_k(y) - C_k(x)| \to 0$ as $Q \downarrow x$ by cont. of C_k.
Thus $\limsup_{Q\downarrow x}\left|\frac{F(Q)}{|Q|} - f(x)\right| \leq (f - C_k)^*(x) + 0 + |f - C_k|(x)$. Let $\varepsilon > 0$, define $E_\varepsilon = \{x : \limsup_{Q\downarrow x}\left|\frac{F(Q)}{|Q|} - f(x)\right| > \varepsilon\}$.
By above, if $x \in E_\varepsilon$, either $(f - C_k)^*(x) > \frac{\varepsilon}{2}$ or $|f - C_k|(x) > \frac{\varepsilon}{2}$, so $E_\varepsilon \subseteq \{(f - C_k)^* > \frac{\varepsilon}{2}\} \cup \{|f - C_k| > \frac{\varepsilon}{2}\}$.
By H-L Inequality, as $f - C_k$ integrable, $|\{(f - C_k)^* > \frac{\varepsilon}{2}\}| \leq \frac{C}{(\varepsilon/2)}\int_{\mathbb{R}^n}|f - C_k|$ (using $C = 2\cdot 5^n$).
Next by Chebyshev's Ineq (Cor 5.12), $|\{|f - C_k| > \frac{\varepsilon}{2}\}| \leq \frac{1}{(\varepsilon/2)}\int_{\mathbb{R}^n}|f - C_k|$, and together these give
$|E_\varepsilon| \leq |\{(f - C_k)^* > \frac{\varepsilon}{2}\}| + |\{|f - C_k| > \frac{\varepsilon}{2}\}| \leq \left(\frac{2C}{\varepsilon} + \frac{2}{\varepsilon}\right)\int_{\mathbb{R}^n}|f - C_k| \to 0$ as $k \to \infty$ by construction of C_k's, and
since $|E_\varepsilon|$ is indep. of k, we get $|E_\varepsilon| = 0$ $\forall \varepsilon > 0$. Let $E = \bigcup_k E_{1/k}$, then $|E| = 0$. For $x \in \mathbb{R}^n \setminus E$,
$x \notin E_{1/k}$ $\forall k$, hence $\limsup_{Q\downarrow x}\left|\frac{F(Q)}{|Q|} - f(x)\right| \leq \frac{1}{k}$ $\forall k \geq 1$, so $\limsup_{Q\downarrow x}\left|\frac{F(Q)}{|Q|} - f(x)\right| = 0 \Rightarrow \lim_{Q\downarrow x}\frac{F(Q)}{|Q|} = f(x)$.
But $|E| = 0$ so this is true for a.e. $x \in \mathbb{R}^n$, as desired. □

- thm 7.11: Lebesgue's Diff thm is still true under the assumption of locally integrable (int on all bounded sets)
- def: Let E m'ble. x is a point of density of E if $\lim_{Q\downarrow x}\frac{|E\cap Q|}{|Q|} = 1$. x is a point of dispersion of E if $\lim_{Q\downarrow x}\frac{|E\cap Q|}{|Q|} = 0$.

* - thm 7.13: Let E m'ble. Almost every point of E is a point of density. [apply thm 7.11 to χ_E]
- Remark: A similar arg shows for ae $x \notin E$, $\lim_{Q\downarrow x}\frac{|E\cap Q|}{|Q|} = 0$

- def: Let $f: \mathbb{R}^n \to \mathbb{R}$ locally int. $x \in \mathbb{R}^n$ is a Lebesgue pt. of f if $\lim_{Q \downarrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|\, dy = 0$
the Lebesgue set of f is the set of Lebesgue pts.
- Remark: If f is cont at x, x is a Lebesgue pt of f.

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- thm 7.15: Let $f: \mathbb{R}^n \to \mathbb{R}$ be locally int. Then a.e. $x \in \mathbb{R}^n$ is a Lebesgue pt of f

Proof: Let $\{r_k\}$ be an enumeration of the rationals. Let Z_k be the subset of $x \in \mathbb{R}^n$ where $\lim_{Q \downarrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|\, dy = |f(x) - r_k|$ FAILS. Since $|f - r_k|$ is locally int, by thm 7.11, equality holds a.e., and so $|Z_k| = 0$. Let $Z = \bigcup_k Z_k$, then $|Z| = 0$ too. Now for any $x \in \mathbb{R}^n$, $k \geq 1$, Q cube (center x), $\frac{1}{|Q|}\int_Q |f(y) - f(x)|\, dy \leq \frac{1}{|Q|}\int_Q |f(y) - r_k|\, dy + \frac{1}{|Q|}\int_Q |r_k - f(x)|\, dy$
$= \frac{1}{|Q|}\int_Q |f(y) - r_k|\, dy + |r_k - f(x)|$. If $x \notin Z$, we have for all k,
$\limsup_{Q \downarrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|\, dy \leq |f(x) - r_k| + |r_k - f(x)| = 2|f(x) - r_k|$. The LHS is indep of k, so we can choose r_k arb. close to $f(x)$, then $\limsup_{Q \downarrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|\, dy = 0$. □

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- thm 7.16 (Fundamental Thm of Calculus): Let $f: [a,b] \to \mathbb{R}$ integrable. Let $F(x) = \int_a^x f(y)\, dy$.
Then $F' = f$ a.e. and in particular, at every Lebesgue pt. of f.

Proof: If $h \neq 0$, $\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} f(y)\, dy - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} (f(y) - f(x))\, dy\right|$
$\leq \frac{1}{|h|}\int_{x-|h|}^{x+|h|} |f(y) - f(x)|\, dy$ which tends to 0 as $|h| \to 0$ for all $x \in [a,b]$ that are Lebesgue pts, i.e., a.e. $x \in [a,b]$. So for a.e. x, $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$. □

- def: A family of cubes K covers E in the Vitali sense if $\forall x \in E$, $\forall \eta > 0$, $\exists Q \in K$ with diameter $< \eta$, > 0, that contains x. [$\exists$ arb. small cube containing x]

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- thm 7.17 (Vitali Covering Lemma): Let $E \subseteq \mathbb{R}^n$, $|E|_e < \infty$. Let K be a collection of cubes covering E in the Vitali sense. Let $\varepsilon > 0$, then there is a sequence of disjoint cubes $\{Q_j\}$ in K s.t. $|E \setminus \bigcup_j Q_j|_e = 0$ and $\sum_j |Q_j| < (1+\varepsilon)|E|_e$.

Proof: $\exists G \supseteq E$ open s.t. $|G| < (1+\varepsilon)|E|_e$. Discard all Q not contained in G. By simple Vitali Lemma, $\exists$ finitely many disjoint $\{Q_j\}_{j=1}^{N_1}$ in K with $\sum_{j=1}^{N_1} |Q_j| \geq \beta |E|_e$.
($\beta = 1/2(5^n)$). Thus, $|E - \bigcup_j^{N_1} Q_j|_e \leq |G - \bigcup_j^{N_1} Q_j|_e = |G| - |\bigcup_r^{N_1} Q_r|$ (as all $Q_j \subseteq G$)
$= |G| - \sum_j^{N_1} |Q_j| < (1+\varepsilon)|E|_e - \beta |E|_e = |E|_e(1 + \varepsilon - \beta)$. So assuming $\varepsilon < \beta/2$ from the start gives $|E - \bigcup_j^{N_1} Q_j|_e < |E|_e(1 - \beta/2)$.

Now repeat this process for $E_1 = E \setminus \bigcup_j^{N_1} Q_j$, which is still covered in the Vitali sense by cubes in K disjoint from $Q_1, \dots, Q_{N_1}$. So by Simple Vitali Lemma, we have $\{Q_j\}_{j=N_1+1}^{N_2}$ disjoint, from K, and disjoint from $Q_1, \dots, Q_{N_1}$, s.t. $|E_1 \setminus \bigcup_{N_1+1}^{N_2} Q_j| < |E_1|_e(1 - \beta/2)$, so
$|E \setminus \bigcup_{j=1}^{N_2} Q_j| = |E_1 \setminus \bigcup_{j=N_1+1}^{N_2} Q_j| < |E_1|_e(1 - \beta/2) < |E|_e(1 - \beta/2)^2$

Continuing, the m^{th} stage disjoint $Q_1, \dots, Q_{N_m}$ in K has $|E - \bigcup_{j=1}^{N_m} Q_j| < |E|_e (1 - \beta/k)^m$. This gives a countable $\{Q_j\}_{j=1}^{\infty}$. For any $m \geq 1$, $|E \setminus \bigcup Q_j| \leq |E \setminus \bigcup^{N_m} Q_j| < |E|_e (1 - \beta/k)^m$, so taking $m \to \infty$, $|E \setminus \bigcup Q_j| = 0$. As all Q_j's are disjoint and in G, $\sum_j |Q_j| = |\bigcup Q_j| \leq |G| < (1+\varepsilon)|E|_e$. □

* • thm 7.21: Monotone fns are differentiable a.e.: let $f : [a,b] \to \mathbb{R}$ be monotone inc (so finite valued).
i) $f'(x)$ exists a.e. in (a,b) ii) f' is m'ble iii) $0 \leq \int_a^b f' \leq f(b) - f(a)$
Remark: for (iii), see Cantor Lebesgue fn.

• Cor 7.23: let $f \in BV[a,b]$. Then f' exists a.e. in $[a,b]$ and $f' \in L_1([a,b])$

• def: let $f : [a,b] \to \mathbb{R}$. We say f is abs. continuous on $[a,b]$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ st. if $\{[a_j, b_j]\}$ are nonoverlapping intervals in $[a,b]$, with $\sum_j (b_j - a_j) < \delta$, then $\sum_j |f(b_j) - f(a_j)| < \varepsilon$.

* • thm 7.27: If f is absolutely continuous on $[a,b]$, then $f \in BV[a,b]$.

• def: let $f : [a,b] \to \mathbb{R}$ be diff a.e.. f is singular if $f' = 0$ a.e. on $[a,b]$

* • thm 7.28: let f be abs. cont and singular on $[a,b]$. Then f is constant on $[a,b]$.

* • thm 7.29: f is abs cont on $[a,b]$ iff both i) f' exists a.e in $[a,b]$ and $f' \in L_1([a,b])$ and
ii) $\forall x \in [a,b]$, $f(x) - f(a) = \int_a^x f'$. [abs cont fns are integrals]

• thm 7.30: let $f \in BV[a,b]$, then $f = g + h$, where g abs cont on $[a,b]$; h is singular on $[a,b]$

• def (σ-algebra): Let S be a set, and $\Sigma \subseteq P(S)$. Σ is a σ-alg if
a) $S \in \Sigma$ b) $E \in \Sigma \Rightarrow E^c \in \Sigma$ c) $E_k \in \Sigma$, $k \geq 1 \Rightarrow \bigcup_k E_k \in \Sigma$

• def: let Σ be a σ-alg on S
a) Let $\phi : \Sigma \to \mathbb{R}$. We say ϕ is an additive set fn on Σ if both 1) $\phi(E)$ is finite $\forall E \in \Sigma$ 2) $\phi(\bigcup E_k) = \sum \phi(E_k)$ disjoint E
b) Let $\mu : \Sigma \to [0, \infty]$. μ is a measure on Σ if both 1) $0 \leq \mu(E) \leq \infty$ $\forall E \in \Sigma$ 2) $\mu(\bigcup E_k) = \sum \mu(E_k)$, disj E_k
We then say (S, Σ, μ) is a measure space.
Remarks: If ϕ is additive, $\phi(E_2 \setminus E_1) = \phi(E_2) - \phi(E_1)$, and taking $E_1 = E_2$ gives $\phi(\emptyset) = 0$.
If μ is a measure, $\mu(\emptyset) = 0$ and $E_1 \subseteq E_2 \Rightarrow \mu(E_1) \leq \mu(E_2)$

* • thm 10.1: let Σ be a σ-alg and ϕ an additive set fn on Σ. Let $\{E_k\}$ monotone in Σ, so $E_k \nearrow E$ or $E_k \searrow E$. Then $\lim_{k \to \infty} \phi(E_k) = \phi(E)$

* • thm 10.2: let ϕ be a nonneg additive set fn, $\{E_k\}$ be any seq of sets in Σ. Then,
$\phi(\liminf E_k) \leq \liminf \phi(E_k) \leq \limsup \phi(E_k) \leq \phi(\limsup E_k)$

- def: let Σ be a σ-alg and ϕ be an additive set fn on Σ. Let $E \in \Sigma$.
  i) the upper variation of ϕ on E is $\overline{V}(E) = \sup\{\phi(A) : A \subseteq E, A \in \Sigma\}$
  ii) the lower variation of ϕ on E is $\underline{V}(E) = -\inf\{\phi(A) : A \subseteq E, A \in \Sigma\}$
  iii) the total variation of ϕ on E is $V(E) = \overline{V}(E) + \underline{V}(E)$

  Remarks: $\phi(\emptyset) = 0$ so $\overline{V}(E), \underline{V}(E) \geq 0$. Also, $E \subseteq F \Rightarrow \overline{V}(E) \leq \overline{V}(F)$ and $\underline{V}(E) \leq \underline{V}(F)$

** - Lemma 10.4 (Subadditivity): Let ϕ be an additive set fn on Σ. Then $\overline{V}, \underline{V}, V$ are subadditive so if $E_k \in \Sigma$, $k \geq 1$, then $\overline{V}(\bigcup_k E_k) \leq \sum_k \overline{V}(E_k)$, same for $\underline{V}, V$.

  Proof: Let $H_1 = E_1$, and for $k \geq 2$, $H_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right)$. Then H_k disjoint in Σ and $\bigcup_k H_k = \bigcup_k E_k$. Next let $A \in \Sigma$, $A \subseteq \bigcup_k E_k$. Then A is the disj union of $\{A \cap H_k\}_{k=1}^{\infty}$.
  By additivity of ϕ, $\phi(A) = \sum_{k=1}^{\infty} \phi(A \cap H_k) \leq \sum_{k=1}^{\infty} \overline{V}(E_k)$ (as $A \cap H_k \subseteq E_k$)
  Taking sup over all $A \subseteq \bigcup_k E_k$ gives $\overline{V}(\bigcup_k E_k) \leq \sum_k \overline{V}(E_k)$. □

- Lemma 10.5: Let ϕ be additive on Σ. Then $\overline{V}, \underline{V}, V$ are finite $\forall E \in \Sigma$.

** - Lemma 10.6: Let ϕ be additive on Σ. If $\{E_k\}$ disjoint in Σ, then $\overline{V}(\bigcup_k E_k) = \sum_k \overline{V}(E_k)$ ($\underline{V}, V$)

  Proof: From subadditivity, we have ($\leq$). Let $\varepsilon > 0$. Choose $A_k \subseteq E_k$, $A_k \in \Sigma$, $k \geq 1$, s.t. $\phi(A_k) > \overline{V}(E_k) - \varepsilon/2^k$. The $\{A_k\}$ are disjoint since $\{E_k\}$ are. Then
  $\sum_k \overline{V}(E_k) < \sum_k \phi(A_k) + \varepsilon/2^k = \left(\sum_k \phi(A_k)\right) + \varepsilon = \phi(\bigcup_k A_k) + \varepsilon \leq \overline{V}(\bigcup_k E_k) + \varepsilon$, and $\varepsilon > 0$ arb.
  so $\sum_k \overline{V}(E_k) \leq \overline{V}(\bigcup_k E_k)$. Similar proof for $\underline{V}$, then $V = \overline{V} + \underline{V}$.

* - thm 10.7: Let ϕ be additive on Σ. Then $\underline{V}, \overline{V}, V$ are additive on Σ.

** - thm 10.8 (Jordan decomposition): Let ϕ be additive on Σ. Then $\phi(E) = \overline{V}(E) - \underline{V}(E)$ $\forall E \in \Sigma$.

  Proof: Let $A, E \in \Sigma$, w/ $A \subseteq E$. By additivity, $\phi(E) = \phi(A) + \phi(E \setminus A) \leq \phi(A) + \overline{V}(E)$.
  Taking inf, $\phi(E) \leq \inf\{\phi(A) : A \subseteq E, A \in \Sigma\} + \overline{V}(E) = -\underline{V}(E) + \overline{V}(E)$.
  In the other dir, $\phi(E) = \phi(A) + \phi(E \setminus A) \geq \phi(A) + \inf\{\phi(B) : B \subseteq E, B \in \Sigma\} = \phi(A) - \underline{V}(E)$.
  Taking sup, $\phi(E) \geq \sup\{\phi(A) : A \subseteq E, A \in \Sigma\} - \underline{V}(E) = \overline{V}(E) - \underline{V}(E)$. □

- def: We say $\{E_k\}$ converges to E if $E = \liminf E_k = \limsup E_k$.

- Cor 10.9: Let $\{E_k\}$ in Σ converging to E. Let ϕ be additive on Σ. Then $\lim_{k} \phi(E_k) = \phi(E)$

* - thm 10.10 (Subadditivity of measure): Let (S, Σ, μ) be a measure space and $\{E_k\}$ msble. Then $\mu(\bigcup_k E_k) \leq \sum_k \mu(E_k)$

* - thm 10.11: Let (S, Σ, μ) be a measure space, $\{E_k\}$ msble.
  i) If $E_k \nearrow E$, $\lim_{k} \mu(E_k) = \mu(E)$. ii) If $E_k \searrow E$, $\mu(E_j) < \infty$ for some j, then $\lim_{k} \mu(E_k) = \mu(E)$.

- def: Let Σ be a σ-alg on S, let $E \in S$, and $f: E \to \overline{\mathbb{R}}$. f is Σ-m'ble (or μ-m'ble) if $\{f > a\} \in \Sigma$ $\forall a$
- thm 10.12: Let (S, Σ, μ) be a measure space, $\{E_k\} \subseteq \Sigma$. i) $\mu(\liminf E_k) \le \liminf \mu(E_k)$
  ii) If $\exists k_0$ st $\mu(\bigcup_{k=k_0}^{\infty} E_k) < \infty$, then $\limsup \mu(E_k) \le \mu(\limsup E_k)$.

* thm 10.13: Let Σ be a σ-alg on S. Let $E \in \Sigma$, $p > 0$, $c \in \mathbb{R}$, $f, g, f_k : E \to \overline{\mathbb{R}}$ m'ble.
  i) $f+g$, cf, f^+, f^-, $|f|^p$, fg are m'ble. If ϕ cont on $\mathbb{R}$, $\phi \circ f$ m'ble. If $f \ne 0$ in E, $1/f$ m'ble.
  ii) $\sup f_k$, $\inf f_k$, $\liminf f_k$, $\limsup f_k$ m'ble. If $\lim f_k$ exists, m'ble.
  iii) χ_F is m'ble iff $F \in \Sigma$. For a simple fn, taking diff vals on $\{E_k\}_{i=1}^{N}$, m'ble iff all E_k m'ble.
  iv) If f nonneg, then $\exists$ nonneg simple fns m'ble $\{f_k\}$ w/ $f_k \nearrow f$.
- def: Let (S, Σ, μ) measure space, $E \in \Sigma$. E is μ-meas 0 if $\mu(E) = 0$. A prop holds μ-a.e. (a.e.(μ)) if it holds outside a set of measure 0.

** thm 10.14 (Egorov): Let (S, Σ, μ) be a measure space and $E \in \Sigma$, $\mu(E) < \infty$. Let $\{f_k\}$ be a seq of m'ble fns on E, finite a.e.. Suppose $\lim f_k = f$ a.e.(μ), and also f is finite valued a.e. Then $\forall \varepsilon > 0$, $\exists A \subseteq E$, $A \in \Sigma$ w/ $\mu(E \setminus A) < \varepsilon$, st $\{f_k\}$ converges unif. to f on A, that is, $\sup_{x \in A} |f_k(x) - f(x)| \to 0$, $k \to \infty$.
  Proof: See thm 4.17 (Egorov for Lebesgue measure)
- def: (S, Σ, μ) m-space, $E \in \Sigma$. Let $f: E \to [0, \infty]$. define the integral of f over E w.r.t. μ by $\int_E f \, d\mu = \sup \sum_j [\inf_{E_j} f] \, \mu(E_j)$ where sup taken over all decomps $E = \bigcup_j^N E_j$, finite, disjoint, m'ble.

* thm 10.16: (S, Σ, μ) m-space, $E \in \Sigma$, $f = \sum_{j=1}^{N} v_j \chi_{E_j}$, $\{E_j\}_j^N$ disj, m'ble, and $v_j \ge 0$.
  Then $\int_E f \, d\mu = \sum_{j=1}^{N} v_j \, \mu(E_j)$.

\+ thm 10.17: Let (S, Σ, μ) m-space, $E \in \Sigma$, $f, g: E \to \overline{\mathbb{R}}$ m'ble. i) If $0 \le f \le g$ in E, then $\int_E f \, d\mu \le \int_E g \, d\mu$
  ii) If $f \ge 0$ in E, $\mu(E) = 0$, then $\int_E f \, d\mu = 0$

* lemma 10.18: (S, Σ, μ) m-space. Let $E \in \Sigma$, $f, g: E \to [0, \infty)$ m'ble, $c \in [0, \infty)$.
  i) $\int_E (f+g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$; $\int_E cf \, d\mu = c \int_E f \, d\mu$. ii) If f simple, $E_1, E_2 \in \Sigma$ disj, then $\int_{E_1 \cup E_2} f \, d\mu = \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu$

* lemma 10.19: (S, Σ, μ) m-space, $E \in \Sigma$, $f_k, g: E \to [0, \infty)$ simple, m'ble. If $f_1 \le f_2 \le \dots$ and $\lim f_k \ge g$, then $\lim_{k \to \infty} \int_E f_k \, d\mu \ge \int_E g \, d\mu$

* thm 10.20 (MCT1): (S, Σ, μ) m-space, $E \in \Sigma$, $f_k: E \to [0, \infty)$ simple, m'ble. If $f_k \nearrow f$ on E then $\lim_{k \to \infty} \int_E f_k \, d\mu = \int_E f \, d\mu$
- thm 10.21 (Linearity of $\int$): (S, Σ, μ) m-space, $E \in \Sigma$, $f, g: E \to [0, \infty)$ m'ble, $c \ge 0$. Then
  i) $\int_E (f+g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$ ii) $E_1, E_2 \in \Sigma$, disjoint then $\int_{E_1 \cup E_2} f \, d\mu = \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu$

- def: (S,Σ,μ) m-space, $E\in\Sigma$, $f: E\to\mathbb{R}$ m'ble. define $\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu$, at least one of one right finite, then $\int_E f\,d\mu$ exists, f is integrable w.r.t. μ if $\int_E f\,d\mu$ finite ($f\in L(E;\mu)$)

* - thm 10.23 (Basic Props): (S,Σ,μ) m-space, $E\in\Sigma$, $f,g: E\to\mathbb{R}$ m'ble, $c\in\mathbb{R}$.
  i) $|\int_E f\,d\mu| \le \int_E |f|\,d\mu$, and $f\in L(E,\mu)$ iff $|f|\in L(E,\mu)$.
  ii) If $|f|\le|g|$ a.e (μ) in E and $g\in L(E,\mu)$, then $f\in L(E,\mu)$, and $\int_E |f|\,d\mu \le \int_E |g|\,d\mu$
  iii) If $f\in L(E,\mu)$, then f finite a.e (μ)
  iv) If $f=g$ a.e and $\int_E f\,d\mu$ exists then $\int_E g\,d\mu$ exists and $\int_E f\,d\mu = \int_E g\,d\mu$.
  v) If $\int_E f\,d\mu$ exists then $\int_E cf = c\int_E f$ exists
  vi) If $f,g\in L(E,\mu)$ then $f+g\in L(E,\mu)$ and $\int_E f\,d\mu + \int_E g\,d\mu = \int_E (f+g)\,d\mu$
  vii) If $f\ge 0$, $m\le g\le M$ on E then $m\int_E f\,d\mu \le \int_E gf\,d\mu \le M\int_E f\,d\mu$

* - thm 10.24: let (S,Σ,μ) m-space, $E\in\Sigma$, $f_k: E\to[0,\infty)$ m'ble. Then $\int_E(\sum_k f_k)\,d\mu = \sum_{k=1}^{\infty}\int_E f_k\,d\mu$.

* - thm 10.25: Let (S,Σ,μ) m-space, $E\in\Sigma$, $E=\bigcup_k E_k$, disj, m'ble, $f: E\to\mathbb{R}$ m'ble. Then $\int_E f\,d\mu = \sum_{k=1}^{\infty}\int_{E_k} f\,d\mu$

- Let (S,Σ,μ) measure space, $E\in\Sigma$, $f, f_k, \phi: E\to\mathbb{R}$ m'ble.

* - thm 10.27 (gen MCT): i) If $f_k\nearrow f$ a.e. on E and $\exists\phi\in L(E,\mu)$ w/ $f_k\ge\phi$ a.e., then $\lim\int_E f_k\,d\mu = \int_E \lim f_k$
  ii) If $f_k\searrow f$ a.e., $\exists\phi\in L(E,\mu)$ w/ $f_k\le\phi$, then $\lim\int_E f_k\,d\mu = \int_E \lim f_k\,d\mu$.

* - thm 10.28 (Uniform Conv): If $\mu(E)<\infty$, $f_k\in L(E,\mu)$, $f_k\xrightarrow{\text{unif}} f$ on E, then $f\in L(E,\mu)$, & $\int_E f_k\,d\mu\to\int_E f\,d\mu$

* - thm 10.29 (Fatou): If $f_k\ge\phi\in L(E,\mu)$ a.e. in E then $\liminf\int_E f_k\,d\mu \ge \int_E(\liminf f_k)\,d\mu$.

* - thm 10.31 (LDCT): If $\lim_{k\to\infty} f_k = f$ a.e. and $\exists\phi\in L(E,\mu)$ w/ $|f_k|\le\phi$ a.e. then $\int_E f_k\,d\mu\to\int_E f\,d\mu$

* - thm 10.32 (Bounded): If $\mu(E)<\infty$, $\lim f_k = f$ a.e., $\exists M>0$ st. $|f_k|\le M$ a.e. then $\int_E f_k\,d\mu\to\int_E f\,d\mu$

- def: (S,Σ,μ) m-space, $E\in\Sigma$. Let ϕ be an additive set fn on Σ.
  i) ϕ is abs. cont. w.r.t. μ on E if $A\subseteq E$, $A\in\Sigma$ gives $\mu(A)=0\Rightarrow\phi(A)=0$
  ii) ϕ is singular on E w.r.t. μ if $\exists Z\subseteq E$ st. $\mu(Z)=0$ and $A\subseteq E\setminus Z\Rightarrow\phi(A)=0$

** - thm 10.33: Let (S,Σ,μ) m-space, $E\in\Sigma$
  i) If ϕ is abs. cont and singular on E w.r.t. μ then $\phi(A)=0$ $\forall A\subseteq E$ m'ble.
  ii) If ϕ,ψ abs cont on E w.r.t. μ, then so are $\phi+\psi$, $\phi\psi$, and $c\phi$ for $c\in\mathbb{R}$. Similarly if ϕ,ψ singular.
  iii) ϕ is abs cont on E w.r.t. μ iff $\overline{V}$, $\underline{V}$ is iff V is
  iv) If $\{\phi_k\}$ additive, abs cont on E w.r.t. μ, $\phi(A)=\lim\phi_k(A)$ $\forall A\subseteq E$ m'ble, then ϕ abs cont. Similarly $\{\phi_k\}$ singular

Proof: i) As ϕ singular, $\exists Z\subseteq E$ st. $\mu(Z)=0$, $\forall E\setminus Z\supseteq B\in\Sigma\Rightarrow\phi(B)=0$. Let $A\subseteq E$ m'ble.
By additivity, $\phi(A)=\phi(A\cap Z)+\phi(A\setminus Z)=\phi(A\cap Z)$ $(A\setminus Z\subseteq E\setminus Z)$.
Now $0\le\mu(A\cap Z)\le\mu(Z)=0$ so $\mu(A\cap Z)=0$, and by abs cont, $\phi(A\cap Z)=0$.

ii) Let $c \in \mathbb{R}$. Let $A \subseteq E$, $A \in \Sigma$, $\mu(A) = 0$. By abs. cont. of ϕ, ψ, $(\phi+\psi)(A) = \phi(A) + \psi(A) = 0$ and $(\phi\psi)(A) = \phi(A)\psi(A) = 0$ and $(c\phi)(A) = c\phi(A) = 0$, so $\phi+\psi$, $\phi\psi$, $c\phi$ are abs cont.

We have by singularity $\exists Z_\phi \subseteq E$ w/ $\mu(Z_\phi) = 0$ and $\exists Z_\psi \subseteq E$ w/ $\mu(Z_\psi) = 0$. Let $Z = Z_\phi \cup Z_\psi$, then $\mu(Z) = 0$. and $Z \subseteq E$. Let $A \subseteq E \setminus Z$, then $A \subseteq E \setminus Z_\phi$ and $A \subseteq E \setminus Z_\psi$. So $(\phi+\psi)(A) = \phi(A) + \psi(A) = 0$, $(\phi\psi)(A) = 0$, $c\phi(A) = 0$, so $\phi+\psi$, $\phi\psi$, $c\phi$ are singular.

iii) $(\Leftarrow)$: If $\overline{V}, \underline{V}$ are abs cont on E wrt μ, by (ii), $\phi = \overline{V} - \underline{V}$ is also.

$(\Rightarrow)$: Sps ϕ abs cont on E wrt μ. If $\mu(A) = 0$ then $\mu(B) = 0$ $\forall B \subseteq A$ m'ble, so $\phi(B) = 0$ $\forall B \subseteq A$ m'ble, then $\overline{V}(A) = \sup\{\phi(B) : B \subseteq A \text{ m'ble}\} = 0$ and $\underline{V}(A) = 0$. So $\overline{V}, \underline{V}$ abs cont on E wrt μ.

iv) Easy from defs

- * thm 10.34: (S, Σ, μ) m-space, $E \in \Sigma$, ϕ additive on E. ϕ is abs cont $\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0$ st $\mu(A) < \delta \Rightarrow |\phi(A)| < \varepsilon$
- * thm 10.35: ϕ is singular on E $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists E_0 \subseteq E$ (E_0 m'ble) w/ $\mu(E_0) < \varepsilon$, $V(E \setminus E_0) < \varepsilon$ ($\phi = \overline{V} - \underline{V}$, $V = \overline{V} + \underline{V}$)
- ** thm 10.36: (Hahn Decomposition): Let $E \in \Sigma$, ϕ additive on E. Then $\exists P \subseteq E$, $P \in \Sigma$ st. i) $\phi(A) \geq 0$ $\forall A \subseteq P$ ii) $\phi(A) \leq 0$ for $A \subseteq E \setminus P$, hence: iii) $\overline{V}(E) = \overline{V}(P) = \phi(P)$ iv) $\underline{V}(E) = \underline{V}(E \setminus P) = -\phi(E \setminus P)$

Proof: for $k \geq 1$, choose m'ble $A_k \subseteq E$ st. $\phi(A_k) > \overline{V}(E) - 1/2^k$. By additivity of $\overline{V}$, $\overline{V}(E \setminus A_k) = \overline{V}(E) - \overline{V}(A_k) < 1/2^k$. By Jordan decomp, $\phi(A_k) = \overline{V}(A_k) - \underline{V}(A_k) \Rightarrow \underline{V}(A_k) = \overline{V}(A_k) - \phi(A_k) \leq \overline{V}(E) - \phi(A_k) < 1/2^k$ (as $A_k \subseteq E$). Let $P = \liminf_{k\to\infty} A_k = \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty A_k$. Then $\forall m$ and $j \geq m$, $\underline{V}(\bigcap_{k=m}^\infty A_k) \leq \underline{V}(A_j) < 2^{-j} \to 0$ as $j \to \infty$. By subadditivity, $\underline{V}(P) \leq \sum_{m=1}^\infty \underline{V}(\bigcap_{k=m}^\infty A_k) = 0$.

By monotonicity, $\forall A \subseteq P$ m'ble, $\underline{V}(A) \leq \underline{V}(P) = 0 \Rightarrow \phi(A) = \overline{V}(A) - \underline{V}(A) \geq 0$, so $\phi \geq 0$ on P (i).

Next, $E \setminus P = E \setminus \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty A_k = \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty (E \setminus A_k)$, then for $m \geq 1$, $\overline{V}(E \setminus P) \leq \overline{V}(\bigcup_{k=m}^\infty E \setminus A_k) \leq \sum_{k=m}^\infty \overline{V}(E \setminus A_k) = 2^{-m+1} \to 0$

By monotonicity, $\forall A \subseteq E \setminus P$, $\overline{V}(A) = 0$ so $\phi(A) = \overline{V}(A) - \underline{V}(A) = -\underline{V}(A) \leq 0$, so (ii) holds.

- def: Let Σ be a σ-alg, $E \in \Sigma$, μ measure on E. We say μ is σ-finite if $E = \bigcup E_k$, $\mu(E_k) < \infty$ $\forall k$.
- * thm 10.38 (Lebesgue Decomposition): Let Σ be σ-alg, $E \in \Sigma$, μ σ-finite, ϕ additive. Then $\exists$ unique decomposition $\phi = \alpha + \sigma$, where α, σ are additive on E; α is abs cont, σ singular. Further, $\exists f \in L(E, \mu)$ st. $\alpha(A) = \int_A f \, d\mu$ $\forall A \subseteq E$ m'ble and $\exists Z$ w/ $\mu(Z) = 0$ st $\sigma(A) = \phi(A \cap E)$ $\forall A \subseteq E$ m'ble
- thm 10.39 (Radon-Nikodym Thm): Let Σ σ-alg, $E \in \Sigma$, μ σ-finite, ϕ additive and abs. cont. Then $\exists f \in L(E, \mu)$ unique st. $\phi(A) = \int_A f \, d\mu$ $\forall A \subseteq E$ m'ble

Remark: We often write $f = d\phi/d\mu$ (Radon-Nikodym derivative)