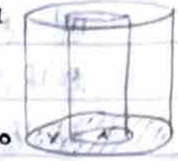
- def: a topology τ on a set X is a collection of subsets of X, called open sets s.t. 1) $\emptyset, X \in \tau$; 2) $U_i \in \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau$ 3) if $U_1, \dots, U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$
- def: $f: X \to Y$ is continuous if $f^{-1}(U) \in \tau$ for all open $U \in \tau$
- def: $C \subseteq X$ is closed if $X \setminus C$ is open
- def: let (X, τ) be a topological space, and $Y \subseteq X$. then Y has the subspace topology $\tau_Y = \{U \cap Y : U \in \tau\}$
- def; let (X, τ) be a space and $q: X \to Y$ is onto (maps are typically taken to be cont.) Y has the quotient topology $\{V \subseteq Y : q^{-1}(V) \in \tau\}$. q is called the quotient map
  - Quotient topology is the largest topology s.t. q is continuous
- def: $f: X \to Y$ continuous bijection, is a homeomorphism if $f^{-1}: Y \to X$ is cont.
- thm: any continuous bijection $f: X \to Y$ when X is compact and Y Hausdorff, is a homeomorphism
- def: X is compact if every open cover has a finite subcover
- def: X is Hausdorff if any two distinct points lie in disjoint open sets
- ex. D^n = closed unit disk in $\mathbb{R}^n$. $D^n \to D^n/\partial D^n \cong S^n$  $\partial D^1 = \{0, 1\}$ gets identified as a pt under quotient map
- def: let X_i be a family of (top.) spaces $i \in I$. $U \subseteq \bigsqcup_{i \in I}$ is open $\Leftrightarrow U \cap X_i$ open $\forall i \in I$ (disj union topology)
- def: a CW/cell complex is a space $\bigcup_{n \geq 0} X^n$ constructed as follows: 1) X^0 is a discrete set 2) X^n is X^{n-1} w/ some points glued to a set of disjoint disks. Formally, let $\{D^n_\alpha\}_\alpha$ be a family of disj. disks. X^n is the quotient $(X^{n-1} \sqcup_\alpha D^n_\alpha)/x \sim \varphi_\alpha(x)$, for each α, $\varphi_\alpha: \partial D^n_\alpha \to X^{n-1}$ 3) $U \subseteq \bigcup_{n \geq 0} X^n$ is open $\Leftrightarrow U \cap X^n$ is open $\forall n \geq 0$.
  - Each X^n is called a n-skeleton
  - $\mathrm{Int}(D^n_\alpha) \hookrightarrow D^n_\alpha \xrightarrow{\text{quotient}} X^n \hookrightarrow X$ is injective, and the image of $\mathrm{Int}(D^n_\alpha)$ is denoted e^n_α, (open) n-cell. [not an open subset] [doesn't include boundary!]
- ex: a CW structure on S^n: $S^0 = \bullet\ \bullet$, $S^1 =$ , $S^2 =$ .
- $\mathbb{R}P^n = S^n / x \sim -x$  (real projective plane), CW struct on $\mathbb{R}P^n$ is induced by CW struct on S^n in prev ex.
- def: a subcomplex A of a CW complex C is a closed subset $A \subseteq C$ that is a union of cells in C. The pair (X, A) is a CW pair.
- def: a pointed/based space is a top space X w/ a distinguished base point $x_0 \in X$
- def: the wedge sum of 2 pointed spaces (X, x_0), (Y, y_0), denoted $X \vee Y$, is the quotient space $X \sqcup Y / \sim$ where $x_0 \sim y_0$ and all other pts are only equiv to themselves.
- def: if X is a CW complex and $X = X^n$, we say X has dimension n.

- def: two (cont.) maps $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$ are homotopic if there is a cont. map $F: X \times I \to Y$ w/ $F(x,0) = f_0(x)$ $F(x,1) = f_1(x)$ $\forall x \in X$. $f_t(x) = F(x,t)$ is a deformation of f_0 to f_1
- def: a space Y is contractible if the identity map $id: Y \to Y$ is homotopic to a constant map (one whose image is a single pt).
  - ▷ If a space is contractible, any two maps are homotopic ($f_0 \to id \to f_1$)
- def: spaces X, Y are homotopy equivalent if there are maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ s.t. $g \circ f: X \to X$ and $f \circ g: Y \to Y$ are homotopic to id_X and id_Y. We say f and g are homotopy equivalences and homotopy inverses of each other.
- def: a deformation retraction of a space X to a subspace A is a (cont.) family of maps $f_t: X \to X$ ($(x,t) \mapsto f_t(x)$) s.t. $f_0(x) = x$ (id) and $f_1(X) \in A$ $\forall x \in X$ and $f_t(a) = a$ $\forall a \in A$, $t \in I$.
  - ★ ∃ contractible space with no deformation retraction to any pt of the space!
- def: given a CW pair (X, A), the quotient space X/A is a CW complex w/ cells of 2 types: 1) the 0-cell A/A 2) each n-cell e^n_α in $X - A$ gives a cell in X/A, the image of e^n_α under $X \to X/A$
- def: let X, Y be CW complexes. $X \times Y$ has CW structure w/ cells $\overset{\text{cell in X}}{e^n_\alpha} \times \overset{\text{cell in Y}}{e^m_\beta}$
  - ★ CW topology is not necessarily the same as product topology! (usually the same)
- def: the cone of a space X is $CX = {X \times I}/{X \times \{0\}}$ 
- def: the suspension of a space X is $SX = {CX}/{X \times \{1\}}$ 
  - ▷ $SD^n = D^{n+1}$ ▷ $SS^n = S^{n+1}$
- def: let $f: X \to Y$. the mapping cylinder is $M_f = {X \times I \sqcup Y}/{(x,1) \sim f(x)}$, and $M_f \overset{\text{homotopy equivalent}}{\simeq} Y$
- def: Suppose $A \xrightarrow{f} X_0$, $A \subseteq X_1$, the space obtained from X_0 by attaching X_1 via f is $X_0 \sqcup_f X_1 = {X_0 \sqcup X_1}/{f(a) \sim a, a \in A}$.
- thm: If (X, A) is a CW pair, and A is contractible, then $X \to X/A$ is a homotopy equiv.

(★) - thm: If (X_1, A) is a CW pair and $A \overset{f}{\underset{g}{\rightrightarrows}} X_0$ are homotopic maps, then $X_1 \sqcup_f X_0$ and $X_1 \sqcup_g X_0$ are homotopy equiv.
- thm: If (X, A) is a CW pair and $A \hookrightarrow X$ (inclusion map) is a homotopy equiv, then one can find a deformation retraction of X onto A.
- ex: any 1-d CW complex is homotopy equiv to a wedge of circles - take a spanning tree and contract that to a pt.

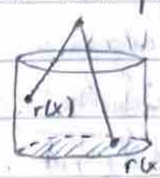
- **def**: let A be a subspace of X. (X, A) has homotopy equivalence property (HEP) if for any pair of $\overset{\text{(cont)}}{\text{maps}}$ $X \times \{0\} \to Y \leftarrow A \times I$ which agree on $A \times \{0\}$ can be extended to $X \times I \to Y$



- **def**: A (cont) map $X \xrightarrow{r} A \subseteq X$ is a retraction if $r(a) = a$ for any $a \in A$. A is a retract of X
- **thm**: (X, A) has HEP iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$
- **ex**: $X = I$, $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, claim that (X, A) does not have HEP
- **thm**: If (X, A) has HEP, then $X \to X/A$ is a homotopy equivalence if A is contractible
- **thm**: any CW pair has HEP

  pf sketch (one cell): $X = D^n$, $A = \partial D^n$

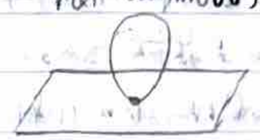
  $D^n \times I \to D^n \times \{0\} \cup \partial D^n \times I$



push pts along line down to $r(x)$, this is continuous

- (★) $X_1 = D^2$, $A = \partial D^2$, $X_0 = \mathbb{R}^2$

$f: \partial D^2 \xrightarrow{id}$ unit circle



$g: \partial D^2 \to \mathbb{R}^2$ (constant map)

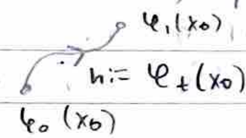
- **def**: a path is a map $f: I \to X$ w/ endpoints $f(0), f(1)$
- **def**: a path homotopy is a homotopy $f_t: I \to X$ s.t. $(x,t) \mapsto f_t(x)$ is cont & $f_t(0) = f_0(0)$, $f_t(1) = f_0(1)$
- **ex**: in $\mathbb{R}^n$ (or any convex subset of $\mathbb{R}^n$), paths w/ same endpts are homotopic:
  - $(1-t) f_0(s) + t f_1(s) = f_t(s)$
- **note**: path homotopy is an equiv. relation
- **def**: product of paths f, g w/ $f(1) = g(0)$ is $f \cdot g(s) = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in (1/2, 1] \end{cases}$

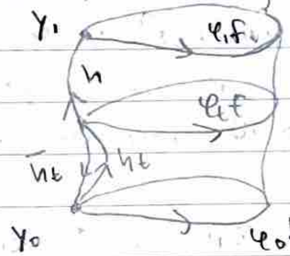
- **def**: the fundamental group $\pi_1(X, x_0)$ is the set of path homotopic classes of loops in X based @ x_0
- **thm**: $\pi_1(X, x_0)$ with the product of paths is a group
  - $\bar{f} = f(1-s)$ denotes the inverse loop/path
- **ex**: If X deformation retracts to x_0 then $\pi_1(X, x_0) = \{1\}$
- **thm**: let $x_0, x_1 \in X$ be joined by a path h ($h(0) = x_0$, $h(1) = x_1$). then the map $\pi_1(X, x_1) \to \pi(X, x_0)$ by $[f] \xrightarrow{\beta_h} [h f \bar{h}]$ is a group isomorphism

- **def**: X is simply connected if for any $x_0, x_1 \in X$, there is a unique path homotopy class from $x_0 \to x_1$
  - X is simply connected iff X is path connected and $\pi(X, x_0) = \{1\}$
- **Prop**: If $\varphi: X \to Y$ map w/ $\varphi(x_0) = y_0$, then the induced map $\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, y_0)$ by $\varphi_*([f]) = [\varphi f]$ is a group homomorphism

- note: 1) $(id_X)_* = id_{\pi_1(X,x_0)}$ 2) $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ w/ $\varphi(x_0)=y_0$, $\psi(y_0)=z_0$, then $(\psi\circ\varphi)_* = \psi_*\varphi_*$.
3) If $\varphi: X \to Y$ is a homeomorphism, then $\varphi_*: \pi_1(X,x_0) \to \pi_1(Y,\varphi(x_0))$ is an isomorphism, inverse $(\varphi^{-1})_*$
- ex: If $r: X \to A$ is a retraction ($A \subset X$, $r|_A = id_A$), then $\pi_1(A,a) \xrightarrow{i_*} \pi_1(X,a) \xrightarrow{r_*} \pi_1(A,a)$
the composition $r_* \circ i_* = (r\circ i)_* = (id_A)_*$ is id, so i_* is 1-1 and r_* is onto.
- note: If $\varphi_t: X \to Y$ is a homotopy and $\varphi_t(x_0) = y_0$ $\forall t$ (indep of t) then in the induced map
$\pi_1(X,x_0) \xrightarrow{\varphi_{t*}} \pi_1(Y,y_0)$, we have $[f] \to [\varphi_t f] = [\varphi_0 f]$ (it's indep of t)
- ex: what if we don't have the condition $\varphi_t(x_0) = y_0$? let $y_0 = \varphi_0(x_0)$, $y_1 = \varphi_1(x_0)$

$$\begin{tikzcd} & \pi_1(Y,y_0) \\ \pi_1(X,x_0) \arrow[ur, "\varphi_{0*}"] \arrow[dr, "\varphi_{1*}"'] & \\ & \pi_1(Y,y_1) \arrow[uu, "\beta_h"'] \end{tikzcd}$$

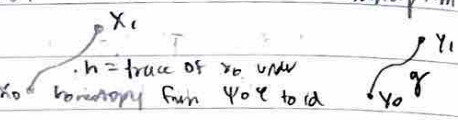




$h_t \varphi_t \bar{h}_t$

Claim: $\beta_h \circ \varphi_{1*} = \varphi_{0*}$
$\beta_h([\varphi_1 f]) = [h \varphi_1 f \bar{h}] \overset{?}{=} [\varphi_0 f]$

- thm: If $\varphi: X \to Y$ is a homotopy equivalence, then $\varphi_*: \pi_1(X,x_0) \to \pi_1(Y,\varphi(x_0))$ is an isomorphism
Pf: $X \underset{\varphi}{\overset{\psi}{\leftrightarrows}} Y$ st. $\varphi\circ\psi \sim id_Y$, $\psi\circ\varphi \sim id_X$.
$\pi_1(X,x_0) \xrightarrow{\varphi_*} \pi_1(Y,\varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X,\psi(y_0)) \xrightarrow{\varphi_*} \pi_1(Y,\varphi(x_1))$
$\varphi(x_0) = y_0$, $\psi(y_0) = x_1$, $\varphi(x_1) = y_1$; β_h, β_γ



$\psi_* \varphi_* = \beta_h$ — isomorphism — β_h 1-1 $\Rightarrow$ φ_* is 1-1
$\varphi_* \psi_* = \beta_\gamma$ — β_γ onto $\Rightarrow$ φ_* is onto $\square$

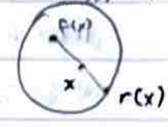
- def: A covering map $p: \tilde{X} \to X$ is a cont. map (surj). st. $\forall x \in X$, $\exists$ open neighborhood containing x, U
with $X \supseteq p^{-1}(U) = \bigsqcup_i U_i$ (disj union of open sets) and $p|_{U_i}: U_i \to U$ a homeomorphism.
- $\tilde{X}$ is a covering space of X, U is said to be evenly covered.
- Claim: let B be I or I^2 and $f: B \to X$ be cont, and $b_0 \in B$. Let $\tilde{X} \xrightarrow{p} X$ be a covering map.
Fix any $\tilde{x}_0 \in p^{-1}(x_0) = p^{-1}(f(b_0))$. Then $\exists$ unique cont. $\tilde{f}: B \to \tilde{X}$ with $\tilde{f}(b_0) = \tilde{x}_0$, $p\tilde{f} = f$. ($\tilde{f}$ is lift)
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ B \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$
- thm: $\pi_1(S^1, 1)$ is infinite cyclic generated by $\omega(s) = e^{2\pi i s}$ ($\pi_1(S^1,1) \cong \mathbb{Z}$)

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Pf: note $\omega(0) = \omega(1) = 1$. Also, $[\omega^n(s)] = [\omega \cdot \ldots \cdot \omega(s)]$. Now take $[f] \in \pi_1(S^1, 1)$,
and let $\tilde{f}: I \to \mathbb{R}$ be the unique lift of $f: I \to S^1$ with $\tilde{f}(0) = 0$. As $p\tilde{f}(1) = f(1) = 1$
and $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, we know $\tilde{f}(1) \in \mathbb{Z}$ (path ends @ integer). Say $\tilde{f}(1) = m$. let $\tilde{\omega}_m$
be the unique lift of ω_m, note $\tilde{\omega}_m(1) = m$. We have the straight line homotopy

$\tilde{f}_t(s) = (1-t)\tilde{f}(s) + t\tilde{\omega}_m(s) \in \mathbb{R}$, so $p\tilde{f}_t$ is a homotopy from $p\tilde{f} = f$ to $p\tilde{\omega}_m = \omega_m$. But $[\omega_m] = [\omega]^m$, so $[f] = [\omega]^m$, and $\pi_1(S_1)$ is cyclic, generated by $[\omega]$. Can $\pi_1(S^1)$ be finite? We WTS $[\omega]^n = 1 \Rightarrow n = 0$.
Consider the homotopy $\omega_n \overset{f_t}{\simeq} 1 = \omega_0$. Lift f_t to $\tilde{f}_t$ with $\tilde{f}_t(0) = 0 \in \mathbb{R}$, so $\tilde{f}_t(0) \in p^{-1}(1) = \mathbb{Z}$.
Next, $p\tilde{f}_t(1) = f_t(1) = 1$, so $\tilde{f}_t(1) \in p^{-1}(1) = \mathbb{Z}$. Look at $\tilde{f}_0(s) = sn$, the lift of ω_n starting at 0, on the other hand, $f_1(s) = \omega_0(s) \Rightarrow \tilde{f}_1(s) = 0$ & we have a continuous homotopy b/w fns w/ same endpts in a discrete setting, so $n = 0$. □

- <u>Thm</u> (Brouwer's Fixed pt thm (for D_2)): Every cont. map $f: D^2 \to D^2$ has an x s.t. $f(x) = x$.



<u>Pf</u>: Suppose $f(x) \neq x$ $\forall x \in D^2$. Let r be the fn that sends $D^2 \to \partial D^2$, note its continuous. Also, $r(x) = x$ for $x \in \partial D^2$ (id on boundary). So r is a retraction, and so $r_*: \pi_1(D_2, x_0) \to \pi_1(\partial D^2, x_0)$ ($x_0 \in \partial D^2$) is surjective. But $\pi_1(D_2, x_0) = 1$ (disk is contractible), and $\pi_1(\partial D^2, x_0) = \mathbb{Z}$, and there cannot be a surjective map from 1 to $\mathbb{Z}$. □

- <u>Thm</u>: $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

<u>Pf</u>: We have projections $X \times Y \xrightarrow{p_X} X$ by $(x,y) \mapsto x$ and likewise for p_Y.
Let $[f] \in \pi_1(X \times Y, (x_0, y_0))$, we can project this to $([p_X f], [p_Y f])$.
On the other hand, given a loop $([\gamma_x(s)], [\gamma_y(s)])$, the loop $[(\gamma_x(s), \gamma_y(s))]$ is mapped onto it (onto). If $[f] = 1$, then so is its projection (1-1). Note p_X, p_Y are homomorphisms, so the product is a homomorphism. □

- <u>Lemma</u>: let $X = \bigcup_\alpha A_\alpha$ be a space, each A_α is open, path connected, and $A_\alpha \cap A_\beta$ is path connected $\forall \alpha, \beta$. Then any loop f based at $x_0 \in X$ is path homotopic (rel x_0) to $f_1 \cdot \ldots \cdot f_n$ s.t. each f_i has image contained in some A_α, based at x_0.

<u>Pf</u>: Let $f: I \to X = \bigcup_\alpha A_\alpha$, then $f^{-1}(A_\alpha)$ is an open cover for I which is compact, so there is a finite subcover $f^{-1}(A_{\alpha_1}), \ldots, f^{-1}(A_{\alpha_n})$. We can get a partition of I, $0 = s_0 < s_1 < \ldots < s_n = 1$, with $[s_i, s_{i+1}] \subseteq f^{-1}(A_{\alpha_{j_i}})$ for every i.
Decompose f into paths by this partition, $f = g_0 \cdot g_1 \cdot \ldots \cdot g_{n-1}$. Then introduce new paths using path connectedness of intersections: $f \simeq (g_0 \bar{h}_1)(h_1 g_2 \bar{h}_2)(h_2 g_3 \bar{h}_3)\ldots$
each $()$ is in A_α and is a loop based @ x_0. □

- <u>Prop</u>: $\pi_1(S^n) = 1$ for $n > 2$.

- Cor: $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^2$ for $n \geq 3$.

Pf: If $f: \mathbb{R}^n \to \mathbb{R}^2$ is a homeomorphism, consider $f|_{\mathbb{R}^n - \{0\}}: \mathbb{R}^n - \{0\} \to \mathbb{R}^2 \setminus \{f(0)\}$, $\mathbb{R}^n - \{0\}$ def. retracts to S^{n-1}, homotopy equiv. But then $1 = \pi_1(S^{n-1}) \cong \pi_1(S^1) = \mathbb{Z}$ $\unicode{x21AF}$ $\square$

- thm (fundamental thm of algebra): every nonconstant (complex) poly has a root.

Pf: Let $p(z) = 1z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$. Assume for sake of contradiction $p(z) \neq 0$ $\forall z$. Let $f_r(s) = (p(re^{2\pi i s}) \cdot |p(r)|) / (|p(re^{2\pi i s})| \cdot p(r))$.
Note $f_r(0) = f_r(1) = 1$, $f_0(s) = 1$, loop is nullhomotopic. We WTS $[f_r] = [\omega_n]$.
define $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \dots + a_0)$, consider "$f_r(s)$ but change p to p_t", it is defined for large enough r (z^n dominates rest of terms). @ $t=1$, we get $f_r(s)$, @ $t=0$, we get $p_0(z) = e^{2\pi i n s} = \omega_n$, so ω_n is nullhomotopic $\Rightarrow n = 0$, but then p is the constant polynomial. $\square$

- thm (Borsuk-Ulam): for any cont map $f: S^n \to \mathbb{R}^n$, there is an $x \in S^n$ s.t. $f(x) = f(-x)$

Pf (some cases): $(n=0)$: $f: S^0 = \{\pm 1\} \to \mathbb{R}^0$ ✓

$(n=1)$: $f: S^1 \to \mathbb{R}$, let $\varphi(x) = f(x) - f(-x)$, is cont. Note $\varphi(x)\varphi(-x) = -(f(x) - f(-x))^2 \leq 0$, suppose φ never vanishes, say $\varphi(a)\varphi(-a) < 0$, then one of $\varphi(a)$ or $\varphi(-a) < 0$, and the other is > 0, so by IVT $\varphi(x) = 0$ for some x in between.

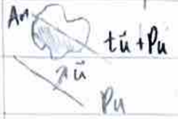
$(n=2)$: $f: S^2 \to \mathbb{R}^2$, Assume $f(x) \neq f(-x)$. Let $g(x) = (f(x) - f(-x)) / |f(x) - f(-x)|$.
Let $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$ be the equator in S^2. Let $h(s) = g(\eta(s))$

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, "p"] & s \arrow[d] \\
S^2 \arrow[rr, bend right, "g"'] & I \arrow[l, "\eta"'] \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & S^1 & (\cos 2\pi s, \sin 2\pi s)
\end{tikzcd}$$

h is homotopic to a constant loop, so h lifts to a loop $\tilde{h}$ in $\mathbb{R}$. So $\tilde{h}(1) = \tilde{h}(0)$. Note g is odd, $g(-x) = -g(x)$. Consider for $s \in [0, 1/2]$,
$h(s + 1/2) = g(\cos(2\pi(s + \frac{1}{2})), \sin(2\pi(s+\frac{1}{2})), 0) = g(-\cos 2\pi s, -\sin 2\pi s, 0)$
$= g(-\eta(s)) = -g(\eta(s)) = -h(s)$. Also,
$h(s+1/2) = p\tilde{h}(s+1/2) = (\cos(2\pi \tilde{h}(s+1/2)), \sin(2\pi\tilde{h}(s+1/2)))$
$\parallel$
$-h(s) = -p\tilde{h}(s) = (\cos(2\pi(\tilde{h}(s) + 1/2)), \sin(2\pi(\tilde{h}(s) + 1/2)))$.
So $\tilde{h}(s+1/2) - \tilde{h}(s) - \frac{1}{2} = \frac{2n}{2} \in \mathbb{Z}$, then $\tilde{h}(s+1/2) - \tilde{h}(s) = q/2$ when q is an odd integer. LHS is cont in s, and RHS is discrete, hence q constant.
Then: $\tilde{h}(\frac{1}{2} + 1/2) = \tilde{h}(1/2) + q/2 = (\tilde{h}(0) + q/2) + q/2$, hence $\tilde{h}(1) - \tilde{h}(0) = q$.
But q is odd, hence nonzero, so we have a contradiction.

• thm (Ham Sandwich): Suppose Borsuk-Ulam holds for dim. $n-1$ (any cont map $S^{n-1} \xrightarrow{f} \mathbb{R}^{n-1}$ has a point $f(x) = f(-x)$). Then for any compact $A_1, \dots, A_n$ in $\mathbb{R}^n$, there is a codimension one plane in $\mathbb{R}^n$ that divides each A_i into two subsets of equal measure.

Proof: For any unit vector u, let P_u be the plane through u that is $\perp$ to u.



Consider a function of t that is the measure of A_n that lies on the same side of $tu + P_u$ as u. Note this is monotone (and continuous!). So there is an interval of t where the function is $\frac{m(A)}{2}$ by IVT. Let t_u be the midpoint of that interval. It's a fact that $u \mapsto t_u$ is cont. Let $f_i(u)$ be the measure of the portion of A_i that lies on the same side of $t_u u + P_u$ as u. Define $f = (f_1, \dots, f_{n-1}) : S^{n-1} \to \mathbb{R}^{n-1}$ by $u \mapsto (f_1(u), \dots, f_{n-1}(u))$. By Borsuk-Ulam, $\exists u_0$ for which $f(u_0) = f(-u_0)$. Note for all u, $f_i(u) + f_i(-u) = m(A_i)$. So $f_i(u_0) = \frac{m(A_i)}{2}$ for every i (up to n) □

• def: Let $\{G_\alpha\}_{\alpha \in A}$ be a family of groups. The free product $\ast_\alpha G_\alpha$ is as a set, the set of reduced words $g_1 \dots g_n$ where each $g_i \in G_{\alpha_i} - \{1\}$, $\alpha_i \neq \alpha_{i+1}$, plus the empty word. As a group, the operation is juxtaposition followed by reduction. The empty word is the identity, and inverse $(g_1 \dots g_n)^{-1} = g_n^{-1} \dots g_1^{-1}$. It's associative :)

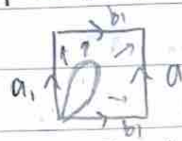
• ex: If each $G_\alpha = \mathbb{Z}$, $\ast_\alpha \mathbb{Z}$ is the free group on the set A. we'll show $\pi_1(S_1 \vee S_1) = \mathbb{Z} \ast \mathbb{Z}$.

• ex: $\mathbb{Z}_2 \ast \mathbb{Z}_2$, generators a, b respectively. Possibilities: $ab\dots a$, $ba\dots a$, $ab\dots b$, $ba\dots b$. We can think of $\mathbb{Z}_2 \ast \mathbb{Z}_2$ as a subgroup of $\mathrm{Isom}(\mathbb{R})$ by $a(x) = -x$, $b(x) = 1-x$ (reflection across $\frac{1}{2}$). $a^2 = id$, $b^2 = id$. This group is also called D_∞ (inf. dihedral)

• Key Property: Any collection of group homomorphisms $\varphi_\alpha : G_\alpha \to H$ extends to a group homomorphism $\ast_\alpha \varphi_\alpha : \ast_\alpha G_\alpha \to H$ that sends the reduced word $g_1 g_2 \dots g_n$ to $\varphi_{\alpha_1}(g_1) \dots \varphi_{\alpha_n}(g_n)$. Further it is the only homomorphism $\ast_\alpha G_\alpha \to H$ that restricts to φ_α on each $G_\alpha \leq \ast_\alpha G_\alpha$

• thm (Van Kampen): Let (X, x_0) be a pointed space and $X = A_1 \cup A_2$, where A_1, A_2 path connected, $x_0 \in A_1 \cap A_2$, and $A_1 \cap A_2$ path-connected. Then the inclusions $A_1 \xrightarrow{i_1} X \xleftarrow{i_2} A_2$ induce a surjection $\pi_1(A_1, x_0) \ast \pi_1(A_2, x_0) \to \pi_1(X, x_0)$ whose kernel is generated by elements $i_{1\ast}(g)\, i_{2\ast}^{-1}(g)$, $g \in \pi_1(A_1 \cap A_2, x_0)$ where $A_2 \xleftarrow{i_2} A_1 \cap A_2 \xrightarrow{i_1} A_1$

- $\pi_1(\mathbb{R}^3 - \bigcirc\bigcirc) \cong \mathbb{Z} * \mathbb{Z}$, $\pi_1(\mathbb{R}^3 - \bigcirc\!\!\bigcirc) \cong \mathbb{Z} \times \mathbb{Z}$
- Lemma: Let $m,n \in \mathbb{N}^+$, $\nu: S^1 \to \mathbb{T}$ (Clifford torus) be $\nu(z) = (z^m, z^n)$, $|z| = 1$. Thm m,n rel prime $\Rightarrow$ ν is 1-1 $\Rightarrow$ ν is homeomorphism on image $\Rightarrow$ $\nu(S^1)$ is a torus knot of type (m,n)
- A_1, A_2 solid tori $= D^2 \times S^1$
- Note: $K_{m,n}$ is unknotted in $\mathbb{T} = S^1 \times S^1 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = \mathbb{R}^2/\mathbb{Z}$
- if $G \cong H$ then $G/C(G) \cong H/C(H)$ ($C(G)$ = center of group)
- if $G \cong H$, then abelianization of groups isomorphic ($G_{ab} = G/[G,G]$)
- thm: Let X be path connected. Let $S^k \overset{\varphi}{\to} X$ be continuous. Let Y be X with D^{k+1} attached along φ ($Y = D^{k+1} \sqcup X$, $x \in \partial D^{k+1} \sim \varphi(x)$). Let $q: X \to Y$, fix $\overset{\varphi(e_1)}{x} \in X$. Then if $k \geq 2$, $q_*: \pi_1(X,x) \to \pi(Y,x)$ is an isomorphism. If $k=1$, then $q_*: \pi_1(X,x) \to \pi_1(Y,x)$ is onto and its kernel is the smallest normal subgroup containing $[\varphi]$
- ex: if X is (path-connected) CW-complex, then $X^2 \overset{i}{\hookrightarrow} X$ is π_1 isomorphism.
- ex: every group $\langle a_\alpha | r_\beta \rangle \cong \pi_1(X^2)$ for some X^2 (2-skeleton)
- $S_g = S^2$ w/ g handles



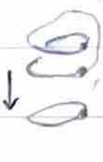
= torus w/ open disk removed
  - $\pi_1(S_1) = \langle a_1, b_1 \mid a_1 b_1 a_1^{-1} b_1^{-1} \rangle$ $= id$
  - $\pi_1(S_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] = 1 \rangle$
  - $\pi_1(S_g) = \langle a_1, b_1, \ldots, a_g b_g \mid \prod_g [a_i b_i] = 1 \rangle$
    - ↳ abelianize to show $\pi_1(S_g) \cong \pi_1(S_h)$ iff $g = h$.

Strat: remove disk, def. retract, find π_1, add disk back in

- lemma (Lifting Lemma): Let $p: \tilde{X} \to X$ be a covering map and homotopy $f_t: Y \to X$, and a cont map $\tilde{f}_0: Y \to \tilde{X}$ with $p\tilde{f}_0 = f_0$. Then there is a unique lift $\tilde{f}_t: Y \to \tilde{X}$ with $\tilde{f}_t|_{t=0} = \tilde{f}_0$

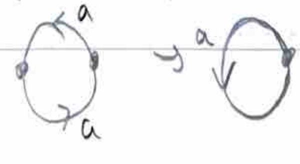
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}_0"] \arrow[r, "f_t"'] & X \end{tikzcd}$$

  ↳ if you can lift homotopy @ $t=0$, you can lift the whole homotopy / extend the lift @ $t=0$.
- thm: Let $p: \tilde{X} \to X$ be a covering map, $\tilde{X}$ path-connected. Then $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$ is injective and the subgroup $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ has index = cardinality of $p^{-1}(x_0)$

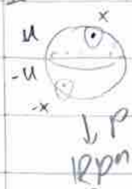


$p: S^1 \to S^1$, $p_* \pi_1(S^1, 1) = n\mathbb{Z} \leq \mathbb{Z}$

$z \mapsto z^n$



- Cor: If $\pi_1(X)=1$ and $p:\tilde{X}\to X$ is a covering, $\tilde{X}$ path-connected, then p is a homeomorphism
- ex: $\mathbb{R}P^n = S^n/x\sim -x$, compute for $n\geq 2$, $\pi_1(\mathbb{R}P^n)$:



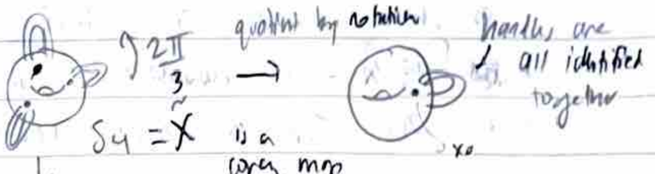
S^n $\pi_1(S^n)=1$ index ≥ 2 $\Rightarrow \pi_1(\mathbb{R}P^n)=\mathbb{Z}_2$

$\downarrow p$ quotient map is a covering map

$\mathbb{R}P^n$

ex: Covering $S_{g\geq 2}\to S_2$ ex. $g=4$

S_2



$S_4=\tilde{X}$ is a covering map

$\downarrow$

$|p^{-1}(x_0)| = g-1 = 3$

$\pi_1(S_2)$ has a subgroup of index $g-1$ isomorphic to $\pi_1(S_g)$

- χ (Euler char) of finite CW complex: #0 cells − #1 cells + #2 cells − ...

0 cells: 1 (all verts identified) $\chi(4g\text{-gon}) = 1-2g+1 = 2-2g$

1 cells: 2 , 2 cells: 1

- thm: Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering map, $f:(Y,y_0)\to(X,x_0)$ be a cont. map. Sps Y path-connected & locally path-connected (any pt has a neighborhood which is path connected). Then a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ exists ($p\tilde{f}=f$) iff $f_*\pi_1(Y,y_0)\subseteq p_*\pi(\tilde{X},\tilde{x}_0)$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

$p\tilde{f}=f$

- ex: Let $Y=S^{n\geq 2}$, $\pi_1(Y)=1$. Any map $Y\to X$ lifts to $\tilde{X}$ (any Y that's simply connected)
- if Y is as in thm, $\pi_1(Y)$ finite, then any $f:Y\to T^n$ (torus) is nullhomotopic

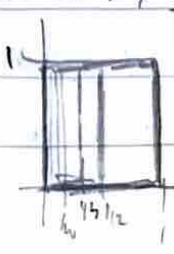
$$\begin{tikzcd} & \mathbb{R}^n \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & T^n \end{tikzcd}$$

- thm: If Y is connected (not a union of two disjoint open sets), then the lift of a map f, $\tilde{f}$, is uniquely determined by $\tilde{f}(y_0)$
- def: two covering maps $(\tilde{X}_1,\tilde{x}_1)\xrightarrow{p_1}(X,x_0)\xleftarrow{p_2}(\tilde{X}_2,\tilde{x}_2)$ are isomorphic if (pointed) $\exists$ homeomorphism $(\tilde{X}_1,\tilde{x}_1)\xrightarrow{h}(\tilde{X}_2,\tilde{x}_2)$ with $p_2h=p_1$, $h(\tilde{x}_1)=\tilde{x}_2$
- thm: the correspondence $p:(\tilde{X},\tilde{x}_0)\to(X,x_0) \rightsquigarrow p_*(\pi_1(\tilde{X},\tilde{x}_0))\leq \pi_1(X,x_0)$ induces a bijection of pointed isomorphism classes of coverings and subgroups of $\pi_1(X,x_0)$ (X is path connected, locally path-connected, any $x\in X$ has nbhd U st. $\pi_1(U,x)\xrightarrow{i_*}\pi_1(X,x)$ is trivial)

Galois correspondence

semi-simply connected

- Lemma: If X is a CW complex, $p:\tilde{X}\to X$ is a covering map, then $\tilde{X}$ has a 'canonical' CW structure whose cells are pre-images of cells in X



Pf: $\star$: the D^n are simply connected, so they lift

p_* injective

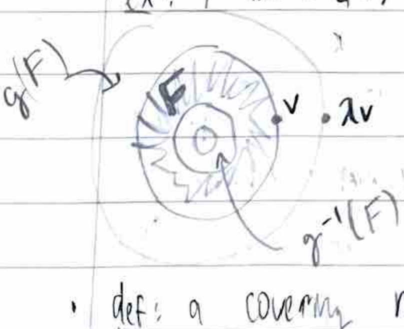
- **thm**: any subgroup of a free group is free.

Pf: let $X = \bigvee_\alpha S^1$, $H \leq \pi_1(X, x_0)$ = free group. By Galois correspondence, $\exists\ \tilde{X} \xrightarrow{p} X$ covering map with $p_*\pi_1(\tilde{X}, \tilde{x}_0) = H$. But by Lemma, $\tilde{X}$ is a 1-dim CW complex, hence is a graph. Take a maximal subtree T. $\pi_1(\tilde{X}) \cong \pi_1(\tilde{X}/T) \simeq \pi_1$(wedge of circles) = free □

- **def**: if $\tilde{X} \xrightarrow{p} X$ covering map and $\pi_1(\tilde{X}) = 1$ then $\tilde{X}$ is called universal cover.
- **def**: let Y be a space and $G \leq \mathrm{Homeo}(Y)$. The G-action on Y ($(g,y) = g(y)$) is covering action if each pt $y \in Y$ has a nbhd U such that $g(U) \cap h(U)$ for all $g \neq h$ in G
- **ex**: $\mathbb{Z}_2$ action on S^n, let $g \in \mathbb{Z}_2$, $g \neq 1$, $g(x) = -x$
- **ex**: $\mathbb{Z}$ action on $\mathbb{R}$, translation ($\mathbb{Z}^n \leq \mathbb{R}^n$)

denoted Y/G

- **def**: G-orbit-space is the quotient $Y/\sim$, $y_1 \sim y_2 \iff \exists g \in G$ w/ $g(y_1) = y_2$
  - ↳ orbit: $G.y = \{g(y) \mid g \in G\}$ (where y can go)
- **thm**: If G-action on Y is a covering action, then $q: Y \to Y/\sim$ quotient map is a covering map. If Y connected, $G = \mathrm{Aut}(q)$
- **def**: If $\tilde{X} \xrightarrow{p} X$ is a covering map, $\mathrm{Aut}(p)$ is the group of homeomorphisms $h: \tilde{X} \to \tilde{X}$ st. $ph = p$ ($G(\tilde{X})$ in the book)
- **ex**: $G = \mathbb{Z}_n$ acts on S^1, $\mathrm{Aut}(q) = G$
- Fundamental domain: subset which has a representative from every orbit
- **ex**: $Y = \mathbb{R}^n - \{0\}$, $G = \mathbb{Z} = \langle g \rangle$, $g(v) = \lambda v$ ($\lambda > 1$ fixed)



$Y/G = S^{n-1} \times S$

$n = 2$: $S^1 \times S^1$ = torus

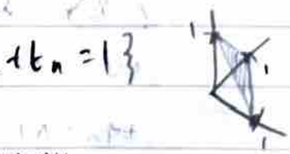
- **def**: a covering map $p: \tilde{X} \to X$ is normal (regular) if $\mathrm{Aut}(p)$ acts transitively on $p^{-1}(x)$ $\forall x \in X$, that is if $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$, $\exists f \in \mathrm{Aut}(p)$ st. $f(\tilde{x}_0) = \tilde{x}_1$ (any base point can go anywhere else)
- **def**: unpointed isomorphism of covering spaces is a homeomorphism $h: X_1 \to X_2$ w/ $p_2 h = p_1$

$$\begin{array}{ccc} X_1 & \xrightarrow{h} & X_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

- thm: Galois correspondence, unpointed: bijection $\left\{\begin{matrix}\text{unpointed isomorphism} \\ \text{classes of covers of } X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy class} \\ \text{of subgroup of } \pi_1(X)\end{matrix}\right\}$
- thm: let $p: \tilde{X} \to X$ covering map, $\tilde{x}$ path-conn (so is X), X locally path conn.
  Let $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$, $p(\tilde{x}_0) = x_0$.
  1) p normal $\iff$ H is normal subgroup of $\pi_1(X, x_0)$
  2) $Aut(p) \cong N(H)/H$, $N(H)$ is largest normal subgroup of G containing H $(= \{g \in \pi_1(X,x_0) \mid gHg^{-1} = H\})$
  3) If p is normal, then $Aut(p) \cong \pi_1(X, x_0)/H$
  4) If $H = 1$, $Aut(p) \cong \pi_1(X, x_0)$ ☆★ computing π_1
- Cor: let G act by covering transformations of Y. If $\pi_1(Y)$, then $G \cong \pi_1(Y/G)$

Homology

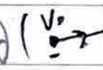
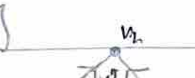
- $\pi_k(X, x_0) = k^{th}$ homotopy group $= [(S^k, s_0), (X, x_0)]$ = pointed homotopic classes of pointed maps $(S^k, s_0) \to (X, x_0)$
- fact: for $k > 1$, if $p: \tilde{X} \to X$ is a covering map, $\pi_k(\tilde{X}) = \pi_k(X)$
- fact: $k \geq 2$, $\pi_k(X, x_0)$ is abelian (hard to compute)
- some homology flavors: simplicial, cellular, singular
- def: standard simplex: $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0,\ t_0 + \dots + t_n = 1\}$
  = smallest convex set in $\mathbb{R}^{n+1}$ containing basis vectors



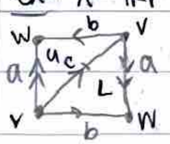
- def: $[v_0 \dots v_n]$ = n-simplex convex hull of $v_0, \dots v_n \in \mathbb{R}^{N \geq n+1}$, if
  $v_0, \dots v_n$ do not lie in a plane of dim $< n$
- note: $e_i \to v_i$ gives a linear homeomorphism $\Delta^n \to [v_0 \dots v_n]$
- def: a face is a convex hull of a subset of the $v_0, \dots v_n$.

face

- def: $\partial[v_0 \dots v_n]$ = boundary = union of proper faces $(\partial(\Delta^0) = \emptyset)$
- def: $\mathring{[v_0 \dots v_n]} = [v_0 \dots v_n] - \partial[v_0 \dots v_n]$ = interior / open n-simplex ($\mathring{\Delta}_n$ = open std. simplex, $\mathring{\Delta}^0 = \Delta^0$)
- def: a Δ-complex (delta complex) is a space X w/ a Δ-complex structure:
  a) collection of cont maps $\sigma_\alpha : \Delta^n \to X$ (n may change w/ α)
  b) $\sigma_\alpha | \mathring{\Delta}_n$ is injective c) each $x \in X$ lies in a unique $\sigma_\alpha(\mathring{\Delta}^n)$ = open cells
  d) $\sigma_\alpha |$ some face = some σ_β. e) $A \subseteq X$ is open $\iff$ $\sigma_\alpha^{-1}(A)$ open in Δ^n $\forall \alpha$.
- ex: S^1: e v, or Δ, $\square$, ... S^2: double of Δ^2 along $\partial \Delta^2$
- Fact: every Δ-complex is a CW complex

- **def**: a simplicial complex structure on X is a Δ-complex structure s.t. if $\sigma_\alpha, \sigma_\beta$ are equal on each vertex of Δ^n, then $\sigma_\alpha = \sigma_\beta$ (multiplicity) $k_\alpha \in \mathbb{Z}$
  ↳ only determined by vertices (dunce of Δ^2 is bad)
- **def**: an n-chain is a formal finite sum $\sum_\alpha k_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is an n-simplex
- **def**: boundary $\partial \sigma_\alpha = \partial [v_0 \ldots v_n] = \sum_{i=0}^{n} (-1)^i [v_0 \ldots \hat{v}_i \ldots v_n]$
  ↑ image of simplex? ↑ remove v_i (notation)
- **ex**: $\partial($  $) = \partial[v_0 v_1] = (-1)^0 [v_1] + (-1)^1 [v_0] = [v_1] - [v_0]$
  ex: $\partial [v_0 v_1 v_2] = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$
  

- **def**: $\Delta^n(X)$ = abelian group of n-chains on X (id = 0, inverse: set $k_\alpha \to -k_\alpha$)
  $\partial_n : \Delta^n(X) \to \Delta^{n-1}(X)$ $\partial_n \sigma_\alpha = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0 \ldots \hat{v}_i \ldots v_n]}$
- **Poincaré lemma**: $\Delta^{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta^n(X) \xrightarrow{\partial_n} \Delta^{n-1}(X)$, $\partial_n \partial_{n+1} = 0$
- **note**: $\Delta^n(X)$ has 2 subgroups: n-cycles $= \ker \partial_n = Z_n(X)$, n-boundaries $= \operatorname{Im} \partial_{n+1} = B_n(X)$
  ↳ lemma says $\ker \partial_n = \operatorname{Im} \partial_{n+1}$
- **def**: homology group: $H_n^\Delta(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$ $\partial_0 = 0$ by convention
- **ex**: $X = S^1$:  $\partial_1 : \Delta^1(X) \to \Delta^0(X)$ by $\partial_1(e) = [v] - [v] = 0$

  $\Delta^{n \geq 2}(X) = 0$ (no other higher dim complexes), $\Delta^1(X) \cong \mathbb{Z}_e$, $\Delta^0(X) \cong \mathbb{Z}_v$
  $H_n^\Delta(X) \cong \Delta^n(X) = \{\mathbb{Z}$ $n = 0, 1$; 0 else
- **ex**: $X = \mathbb{R}P^2$ $\Delta^{n \geq 3}(X) \xrightarrow{\partial_3} \Delta^2(X) \xrightarrow{\partial_2} \Delta^1(X) \xrightarrow{\partial_1} \Delta^0(X) \xrightarrow{\partial_0} 0$
  $\Delta^{n\geq 3}(X) = 0$, $\Delta^2(X) = \mathbb{Z}_U \oplus \mathbb{Z}_L$, $\Delta^1(X) = \mathbb{Z}_a \oplus \mathbb{Z}_b$, $\Delta^0(X) = \mathbb{Z}_v \oplus \mathbb{Z}_w$
  

  $\partial a = w - v$, $\partial b = w - v$, $\partial c = v - v = 0$
  $\partial U = c + b - a$ $\partial L = c + a - b$ (either way doesn't matter)
  ∂_2 is 1-1: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly indep so $\partial_2(mU + nL) = 0$ iff $m = n = 0$.
  $\Rightarrow H_2^\Delta(X) = \ker(\partial_2) / \operatorname{Im}(\partial_3) = 0/? = 0$
  $\operatorname{Im}(\partial_1) = \operatorname{span}(w - v) \Rightarrow H_0^\Delta(X) = \frac{\ker(\partial_0)}{\operatorname{Im}(\partial_1)} = \Delta^0(X) / \operatorname{span}(w-v) = \mathbb{Z}_v \oplus \mathbb{Z}_w / \operatorname{span}(w-v) \cong \mathbb{Z}$ $\langle v, w \mid v = w \rangle$
  $\ker(\partial_1) = \operatorname{span}(c, a - b)$ (basis for $\Delta^1(X)$ can be $a, a-b, c$ $\mathbb{Z}_a \oplus \mathbb{Z}_{a-b} \oplus \mathbb{Z}_c$
  in $H_1^\Delta(X)$, $-c = a - b$ $H_1^\Delta(X) = \frac{\ker(\partial_1)}{\operatorname{Im}(\partial_2)} = \frac{\operatorname{span}(c)}{2c} = \mathbb{Z}_2$
- **def**: a singular n-simplex is a continuous map $\sigma : \Delta^n \to X$
- **def**: singular chains $= C_n(X)$ free abelian group on set of singular simplices
- **def**: an n-chain is a finite sum $\sum_\alpha n_\alpha \sigma_\alpha$ $n_\alpha \in \mathbb{Z}$, σ_α singular n-simplex
- **def**: boundary: $\partial_n\left(\sum_\alpha n_\alpha \sigma_\alpha\right) = \sum_\alpha n_\alpha \partial(\sigma_\alpha)$, $\partial \sigma_\alpha = \sum_{i=0}^{n} (-1)^n \sigma|_{[e_0 \ldots \hat{e}_i \ldots e_n]}$

$\mathrm{Im}\,\partial_{n+1} \subseteq \ker \varepsilon$ (exact)

- singular homology: $H_n(X) = \ker \partial_n / \mathrm{Im}\,\partial_{n+1}$
- thm: If X is path connected then $H_0(X) \cong \mathbb{Z}$
  - $\hookrightarrow$ $H_0(X)$ = free abelian on # path comp. of X $\forall n$
- thm: If $X = \bigcup_\beta X_\beta$ is the decomposition into path connected comps, then $H_n(X) \cong \bigoplus_\beta H_n(X_\beta)$
- thm: If $X = \{pt\}$ then $H_n(X) = \begin{cases} 0 & n>0 \\ \mathbb{Z} & n=0 \end{cases}$
- def: reduced singular homology $\tilde{H}_n(X) = \ker \partial_n / \mathrm{Im}\,\partial_{n+1} \cong H_n(X)$ if $n>0$,

$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, $\sum n_\alpha \sigma_\alpha \to \sum n_\alpha$: $\tilde{H}_0(X) = \ker(\varepsilon) / \mathrm{Im}\,\partial_1$

$$\begin{array}{ccc} \ker \varepsilon & \to & \tilde{H}_0(X) \cong \ker\varepsilon / \mathrm{Im}\,\partial_{n+1} \\ \downarrow \text{inclusion} & & \downarrow \\ C_0(X) & \to & H_0(X) \cong C_0(X)/\mathrm{Im}\,\partial_{n+1} \\ \downarrow \varepsilon & & \downarrow \bar{\varepsilon} \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

$\ker(\bar{\varepsilon}) = \ker\varepsilon / \mathrm{Im}\,\partial_1$

$\Rightarrow \tilde{H}_0(X) = \ker(\bar{\varepsilon} : H_0(X) \to \mathbb{Z})$

$\boxed{H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}}$

(if X path-conn $\tilde{H}_0(X)$ is trivial)

here

$$\begin{array}{ccccc} H \trianglelefteq K & \to & G & \xrightarrow{\varphi} & G/K \\ \text{(norm)} \downarrow & & \downarrow & & \downarrow \\ K/H & \to & G/H & \to & (G/H)/(K/H) \end{array}$$

$\tilde{H}_n(pt) = 0 \quad \forall n$

diff ∂

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial_{n-1}} & C_n(X) \\ \downarrow f_\# & & \downarrow f_\# \\ C_n(Y) & \xrightarrow{\partial_{n-1}} & C_{n-1}(Y) \end{array}$$

- Lemma: If $f: X \to Y$ cont then the induced map $f_\#: C_n(X) \to C_n(Y)$ given by $f_\#(\sum n_\alpha \sigma_\alpha) = \sum n_\alpha f_\#(\sigma_\alpha)$ commutes w/ ∂ (square commutes)
- def: a chain complex (C_*, ∂) is a seq. of abelian group homomorphisms $\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots$ w/ $\partial_n \partial_{n+1} = 0$ $\forall n$. Homology of $(C_*, \partial) = H_n(C) = \ker \partial_n / \mathrm{Im}\,\partial_{n+1}$
- def: a chain map $(C_*, \partial) \to (C'_*, \partial')$ is a seq of abelian group homomorphisms $h_n : C_n \to C'_n$ w/ square commuting:

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ h_n \downarrow & & \downarrow h_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

- Lemma: any chain map $h: (C_*, \partial) \to (C'_*, \partial')$ induces a homomorphism $h_*: H_n(C_*) \to H_n(C'_*)$
- $\boxed{\text{Corollary: } f: X \to Y \text{ induces } f_*: H_n(X) \to H_n(Y)}$
- Corollary: If $f: X \to Y$ is homotopy equivalence then $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.
- Lemma: two homotopic maps $f, g: X \to Y$ induce the same map on singular homology $f_* = g_*$
- def: two chain maps $h_1, h_2 : (C, \partial) \to (C', \partial')$ are chain homotopic if $\exists$ sequence of group homomorphisms $P: C_n \to C'_{n+1}$ with $\partial' P + P\partial = h_1 - h_2$
- Lemma: chain homotopic maps h_1, h_2 induce the same map on homology, ($h_{1*} = h_{2*}$)

$$\begin{array}{ccccccc} \ldots \to & C_{n+1} & \to & C_n & \xrightarrow{\partial} & C_{n-1} & \to \ldots \\ & & \swarrow P & & \swarrow P & & \\ \to & C'_{n+1} & \xrightarrow{\partial'} & C'_n & \to & C'_{n-1} & \to \ldots \end{array}$$

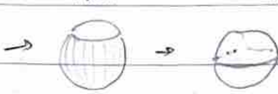
boundary of prism ↓   sides of prism ↓   top ↓   bottom ↓

$\partial P + P\partial = f_{1\#} - f_{0\#}$

- def: A sequence of group homomorphisms is exact $\dots \to A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} A_{n-2} \to \dots$ if $\operatorname{Im}(\alpha_n) = \ker(\alpha_{n-1})\ \forall n$
- ex: chain complex exact $\Leftrightarrow H_n(C) = 0\ \forall n$
- ex: short exact sequence: $H \trianglelefteq G : 1 \to H \to G \to G/H \to 1$
- ex: $0 \to A \xrightarrow{\alpha} B$ is exact $\Leftrightarrow \alpha$ is injective
- ex: $A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow \alpha$ is surjective
- ex: $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow \alpha$ is an isomorphism
- def: (long exact seq of a pair): (X,A) pair, A nonempty closed subset of a space X s.t. A is a retract of some nbhd of A in X then there is exact seq: $\dots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \dots \to \tilde{H}_0(X/A) \to 0$

quotient $q: X \to X/A$

$i: A \to X$ inclusion

- ex: $(X,A) = (D^n, \partial D^n)$ then (boundary to pt) 

$X/A \simeq S^n$

$\tilde{H}_n(D^n) = 0$ $\quad \dots \to \tilde{H}_n(A) \to \underbrace{\tilde{H}_n(X)}_{0} \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \underbrace{\tilde{H}_{n-1}(X)}_{0} \to \dots$

$\Rightarrow \underbrace{\tilde{H}_i(X/A)}_{S^n} \cong \tilde{H}_{i-1}(S^{n-1})$

$\tilde{H}_0(\overbrace{D^n/\partial D^n}^{S^n}) = \begin{cases} 0 & n \geq 1 \to \text{connected} \\ \mathbb{Z} & n = 0 \to \text{two pts} \end{cases}$

- Brouwer's fixed point thm: any cont $f: D^n \to S$ has a fixed pt
- Corollary: any square real matrix A w/ nonneg entries has a nonnegative eigenvector.
- relative singular chain group ($A \subseteq X$ top space)

$0 \to C_n(A) \to C_n(X) \to \underbrace{C_n(X)/C_n(A) = C_n(X,A)}$ $\to 0$

$\partial \downarrow \qquad \downarrow \partial \qquad \downarrow \partial: \partial(C + C_n(A)) = \partial(C) + C_{n-1}(A)$

$0 \to C_{n-1}(A) \to C_{n-1}(X) \to C_{n-1}(X)/C_{n-1}(A)$

short exact sequence

this ∂ satisfies $\partial^2 = \partial_n \partial_{n+1} = 0$

- relative homology: $H_n(X,A) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$
- induces long exact sequence in homology

$$\begin{array}{ccccc} & 0 & & 0 & 0 \\ & \downarrow & & \downarrow & \downarrow \\ \dots \to & C_n(A) & \to \dots & C_0(A) \xrightarrow{\varepsilon} & \mathbb{Z} \to 0 \\ & \downarrow & & \downarrow & \downarrow \wr \\ \to & C_n(X) & \to \dots & C_0(X) \xrightarrow{\varepsilon} & \mathbb{Z} \to 0 \\ & \downarrow & & \downarrow & \downarrow \\ \dots \to & C_n(X,A) & \to \dots & C_0(X,A) \to & 0 \to 0 \\ & \downarrow & & \downarrow & \downarrow \\ & 0 & & 0 & 0 \end{array}$$

columns are short exact sequences

(case $A \neq \emptyset$): $\tilde{H}_n(X,A) = H_n(X,A)$

case: $A = \emptyset$:

$\Rightarrow C_n(A) = 0$, $H_n(X, \emptyset) = H_n(X)$

- ex: $H_i(D^n, \partial D^n)$

$(n=0)$: $\partial D^0 = \phi$, $H_i(D^n, \partial D^n) = H_i(D^0) = H_i(\{pt\}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$

$(n>0)$: $\partial D^n \neq \phi$, $H_i(D^n, \partial D^n) \cong \tilde{H}_i(D^n, \partial D^n)$ $\qquad \tilde{H}_{i-1}(S^{n-1})$

$\underbrace{\tilde{H}_i(D_n)}_{0} \to \tilde{H}_i(D^n, \partial D^n) \to \tilde{H}_{i-1}(\partial D^n) \to \underbrace{\tilde{H}_i(\partial D^n)}_{0}$ (contractible) $\Rightarrow \tilde{H}_i(D^n, \partial D^n) \cong \tilde{H}_{i-1}(\partial D^n)$

- ex: fix $p \in X$: $\underbrace{\tilde{H}_i(p)}_{0} \to \tilde{H}_i(X) \to \tilde{H}_i(X, p) \to \underbrace{\tilde{H}_{i-1}(p)}_{0}$ contractible $\Rightarrow \tilde{H}_i(X) \cong \tilde{H}_i(X, p)$

unreduced homology?: $\underset{\mathbb{Z}}{\overset{}{H_0(p)}} \to H_0(X) \to H_0(X, p) \to 0$ (more complicated w/ $\mathbb{Z}$)

- ex: (X, A): $X = S^n$, A homeomorphic to S^m $(m<n)$ (hint: $n=3, m=1$)

$\tilde{H}_i(A) \to \tilde{H}_i(X) \to \tilde{H}_i(X, A) \to \tilde{H}_{i-1}(A)$ $\qquad \left(\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & n=i \\ 0 & i\neq 0 \end{cases}\right)$

- $i \notin \{n, m+1\} \Rightarrow \tilde{H}_i(X) = 0$, $\tilde{H}_{i-1}(A) = 0 \Rightarrow \tilde{H}_i(X, A) = 0$.
- $i = m+1 = n \Rightarrow 0 \to \mathbb{Z} \to \tilde{H}_n(X, A) \to \mathbb{Z} \to 0 \Rightarrow \tilde{H}_n(X, A) = \mathbb{Z} \oplus \mathbb{Z}$
- $i = n \neq m+1 \Rightarrow \underset{0}{\tilde{H}_n(S^m)} \to \underset{\mathbb{Z}}{\tilde{H}_n(S^n)} \to \underset{\Rightarrow \mathbb{Z}}{\tilde{H}_n(S^n, A)} \to \underset{0}{\tilde{H}_{n-1}(S^m)}$
- $i = m+1 \neq n \Rightarrow \underset{0}{\tilde{H}_{m+1}(S^n)} \to \underset{\Rightarrow \mathbb{Z}}{\tilde{H}_i(S^n, A)} \to \underset{\mathbb{Z}}{\tilde{H}_m(S^m)} \to \underset{0}{H_m(S^n)}$ ($m<n$ used)

- thm: (Invariance of domain): If U is open in $\mathbb{R}^n$, V open in $\mathbb{R}^m$, U, V homeomorphic, then $m=n$.
- Corollary: S^n has no proper subset A homeomorphic to S^m $(n \leq m)$
- ex: $\mathbb{R}^{n \neq m}$ is not homeomorphic to $\mathbb{R}^m$

$\mathbb{R}^m \xrightarrow{h \text{ homeo}} \mathbb{R}^n$, fix $p \in \mathbb{R}^m$, $\underset{S^{m-1}}{\underset{\simeq}{\mathbb{R}^m - \{p\}}} \xrightarrow{\text{homeo}} \underset{S^{n-1}}{\underset{\simeq}{\mathbb{R}^n - h(p)}}$ $\quad \underset{H_{n-1}(S^{n-1})}{\overset{\mathbb{Z}}{\parallel}} \neq \underset{H_{n-1}(S^{m-1})}{\overset{0}{\parallel}}$

- (excision): let $Z \subseteq A \subseteq X$, sps $\bar{Z} \subseteq \mathring{A}$ (closure ⊆ interior).

Then the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphism on homology

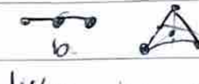
- ex for $\bar{Z} \subseteq \mathring{A}$ necessary: $X = [0,1]$, $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$H_0(X-A, \phi) = \bigoplus_{\alpha \in \mathbb{Z}} \mathbb{Z}$ (one $\mathbb{Z}$ for each path-conn comp of $X-A$) → not cyclic

$\underset{\mathbb{Z}}{H_0(X)} \to H_0(X, A) \to 0$

↪ cyclic b/c surjective image of $\mathbb{Z}$

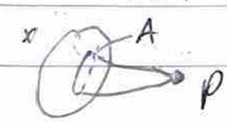
- (excision equivalent): $A \subset X \supset B$ $\mathring{A} \cup \mathring{B} = X$. Then $(B, A\cap B) \to (X, A)$ induces an isomorphism on homology
  $\triangleright$ $B = X - Z$, so $A \cap B = A - Z$ and $(B, A\cap B) = (X-Z, A-Z)$
- Lemma: let $\mathcal{U} = \{U_i\}_i$ and $X = \bigcup_i \mathring{U}_i$. Define:
  $C_n^{\mathcal{U}}(X) = \{ \sum_\alpha n_\alpha \sigma_\alpha \in C_n(X) :$ image of each σ_i lies in some $U_i \}$.
  $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)$ so $(C_n^{\mathcal{U}}, \partial)$ is a chain complex. And $C_n^{\mathcal{U}}(X) \xrightarrow{i} C(X)$ is a chain map.
  Thm $i_* : H_n^{\mathcal{U}}(X) \to H_n(X)$ is an isomorphism
- barycenter of a simplex $[b_0, \dots, b_n] \Rightarrow b = (b_0 + b_1 + \dots b_n)/(n+1)$



- thm: (X,A) a good pair, then $q : (X,A) \to (X/A, A/A)$ induces isomorphism on homology. $H_n(X,A) \cong \tilde{H}_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ (homology rel pt is same)
- long exact seq of triple $B \subseteq A \subseteq X$

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0 \quad \text{inclusion commutes w/ boundary}$$
$$C_n(A)/C_n(B) \to C_n(X)/C_n(B) \to C_n(X)/C_n(A) = \left(C_n(X)/C_n(B)\right) \Big/ \left(C_n(B)/C_n(A)\right)$$

$$\to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \dots$$



- Claim: $H_n(X,A) \cong \tilde{H}_n(X \cup CA)$ for all pairs
  pf: $\tilde{H}_n(CA) \to \tilde{H}_n(X \cup CA) \to \tilde{H}_n(X \cup CA, CA) \to \tilde{H}_{n-1}(CA)$
  $\tilde{H}_n(CA) = 0$, $\tilde{H}_{n-1}(CA) = 0$; isomorphism; $\cong$ excise pt
  $\tilde{H}_n(X \cup CA - p, CA - p) = H_n(X,A)$
- thm: $i_\alpha : X_\alpha \to \bigvee_\beta X_\beta$, (X_α, x_α) a good pair. Then $\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha)$ isomorphism.
- local homology of X at $x \in X$ is $H_n(X, X - \{x\})$
  If $x \in U$ open (and $\{x\}$ closed), can excize $X - U = Z$
  $\Rightarrow H_n(X, X - \{x\}) \cong H_n(X - Z, X - \{x\} - Z) \cong H_n(U, U - \{x\})$
  $\star$ only care abt small nbhd of x
- ex: n-manifold X $x \in X$: $H_i(X, X - \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \overset{\partial}{\cong} \tilde{H}_{i-1}(\mathbb{R}^n - \{0\})$
  $\cong \tilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & n=i \\ 0 & \text{else} \end{cases}$
- X n-manifold w/ boundary: ∂X disjoint from Int X? local homology $\neq$
  $x \in \partial X$ $H_i(X, X - \{x\}) = H_i(\mathbb{R}^{n-1} \times [0,\infty), \mathbb{R}^{n-1} \times [0,\infty) - \{0\}) \overset{\partial}{\cong} \tilde{H}_{i-1}(\mathbb{R}^{n-1} \times [0,\infty) \setminus \{0\}) = 0$

$S^n * S^m = S^{n+m+1}$

$U - \{x\}$

U contractible, $\tilde{H}_i(U) = 0$

- ex: graph: $\tilde{H}_i(X, X - \{x\}) \simeq H_i(U, U - x) \overset{\partial}{\cong} H_{i-1}(U - x)$
- Thm: X Δ-complex, A (empty?) subcomplex of X. Then the chain map $\Delta_n(X,A) \to C_n(X,A)$ that sends each n-simplex $[v_0, \dots, v_n]$ to $\sigma: [e_0, \dots, e_n] \to [v_0, \dots, v_n]$ ($e_i \mapsto v_i$) induces an isomorphism $H_n^\Delta(X,A) \cong H_n(X,A)$

the canonical chain map gives isomorphism

- five lemma:

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{h} & D & \xrightarrow{\ell} & E \\ \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \downarrow\delta & & \downarrow\epsilon \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{h'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

rows are exact seqs

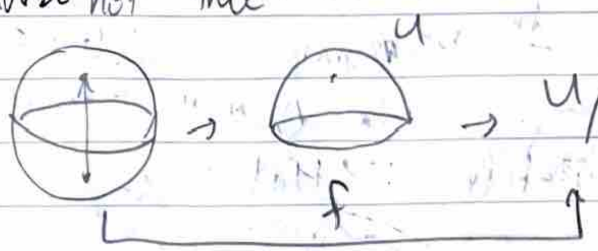
a) γ is 1-1 if β, δ are 1-1 and α onto
b) γ is onto if β, δ are onto and ε is 1-1
$\Rightarrow$ If $\alpha, \beta, \delta, \varepsilon$ isomorphism then so is γ.

- degree Theory:

Maps $f: S^n \to S^n$ induce homomorphism on homology. $f_*: H_n(S^n) \circlearrowleft$, $H_n(S^n) = \mathbb{Z}$

$\mathbb{Z} \to \mathbb{Z}$ call degree of f = $\deg(f) = k$
$1 \mapsto k$ 1) $\deg(id) = 1$ 2) homotopic maps have same deg
$m \mapsto mk$ 3) if f is not surj, $\deg(f) = 0$ $\tilde{H}_n(S^n) \xrightarrow{f_*} \tilde{H}_n(S_n)$
converse not true:

$\tilde{H}_n(S^n - \{pt\})$ inclusion
$= 0$
(contractible)



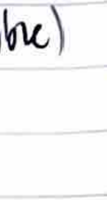
f is surj but $\deg(f) = 0$

4) $\deg(f \circ g) = \deg f \cdot \deg g$
5) f is homotopy equivalence $\Rightarrow \deg f = \pm 1$ (f homeomorphism in particular)
$S^n \underset{g}{\overset{f}{\rightleftarrows}} S^n$ $\deg f \deg g = \deg(g \circ f) = \deg(id) = 1$
6) r = reflection across $x_1, \dots, x_n$ plane

$$r\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ -x_{n+1} \end{bmatrix}$$

S^n = double of Δ^n (along boundary)

$H_n^\Delta(S^n) = \underset{\Delta_u^n - \Delta_L^n}{\mathbb{Z}}$

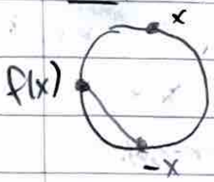


induced chain map

$r_\#(\Delta_u^n - \Delta_L^n) = \Delta_L^n - \Delta_u^n \Rightarrow \deg(r) = -1$

7) $a(x) = -x$ antipodal map
but a = composite of $n+1$ reflections (each coordinate)
$\Rightarrow \deg(a) = (-1)^{n+1}$

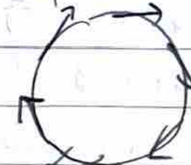
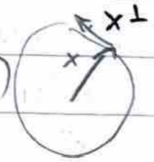
- thm: let $f: S^n \to S^n$ have no fixed pt. Then $\deg(f) = (-1)^{n+1}$



the segment $[f(x), -x]$ doesn't pass the origin
so straight line homotopy between f and a

- thm (Hairy ball): S^n has a cont. nonzero tangent vector field $\Leftrightarrow$ n is odd

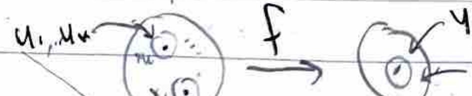
($\Leftarrow$):

$(n=1)$  $(n=2k-1)$ 

$x = (x_1, x_2, \dots, x_{2k-1}, x_{2k})$
$x^\perp = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$
$x \cdot x^\perp = 0$

- thm: let $G \leq \mathrm{Homeo}(S^n)$ that acts freely ($\forall g \in G$, if $g(x) = x$, then $g = 1$)
If n is even, then $G \cong \mathbb{Z}_2$ or $\{1\}$

- Assume: $f: S^n \circlearrowleft$, $\exists y \in S$ with $f^{-1}(y)$ finite set
Want to calculate $\deg(f)$
$(n>0)$



$$H_n(U_i, U_i - x) \xrightarrow{f_*} H_n(V, V-y) \qquad f(U_i) \subseteq V$$

$\cong \mathbb{Z}$; excision $\cong$; excision $\cong$

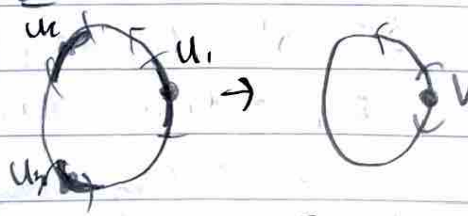
$$H_n(S^n, S^n - x_i) \xleftarrow{i} H_n(S^n, S^n - f^{-1}(y)) \xrightarrow{f_*} H_n(S^n, S^n - y)$$

$n>0$ $\cong$ contractible; i_*; $n>0$

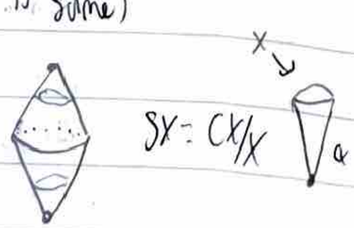
$$H_n(S^n) \xrightarrow{f_*} H_n(S^n) \cong \mathbb{Z}$$

- Lemma: $\deg(f) = \sum_{i=1}^{k} \deg f|_{x_i}$ → local degree $H_n(U_i, U_i - x) \to H_n(V, V-y)$

- ex: $f: S^1 \circlearrowleft$ $z \mapsto z^k$



$\deg f|_{x_i} = 1$ (orientation is same)
so $\deg f = k$



- Lemma: If $f: S^n \circlearrowleft$, $Sf: S^{n+1} \circlearrowleft$ then $\deg(Sf) = \deg(f)$ (S → suspension)

- ex:  $S^n \to S^n / S^n - \cup U_i = \vee S^n \to S^n$ (projection)

$\deg(f) = \sum^{k} \deg f|_{x_i} = k$

- def: an n-manifold M is orientable if there is a consistent choice of a generator in $H_n(M, M-x) \simeq \mathbb{Z}$ $\forall x \in M$

consistent:

$$H_n(\mathbb{R}^n, \mathbb{R}^n - B) \simeq \mathbb{Z}$$

$$\mathbb{Z} \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - x) \qquad H_n(\mathbb{R}^n, \mathbb{R}^n - y) \simeq \mathbb{Z}$$

same generator when going up

- cellular chain groups: $C_n^{CW}(X) = H_n(X^n, X^{n-1})$ = free abel. on n-cells

$$
\begin{array}{ccccc}
 & & & & H_n(X^{n+1}, X^n) = 0 \\
 & & & \nearrow & \\
0 = H_n(X^{n-1}) & & & H_n(X^{n+1}) \simeq H_n(X) & \\
 & \searrow & \nearrow & & \\
 & & H_n(X^n) & & \\
 & \nearrow & & \searrow & \\
H_{n+1}(X^{n+1}, X^n) & & \xrightarrow{d_{n+1}} & & H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_n(X^{n-1}, X^{n-2})
\end{array}
$$

generated by n-cells

$C_n^{CW}(X) = \bigoplus_{n\text{-cells}} H_n(\Delta^n, \partial\Delta^n)$

$H_{n+1}(X^{n+1}, X^n) = C_{n+1}^{CW}(X)$

$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{\text{inclusion}} H_n(X^{n-1}, X^{n-2})$

$H_{n-1}(X^{n-2}) = 0$

$d_{n+1} d_n = 0$

les $(H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}))$

$H_n^{CW}(X)$ = homology of $(C_n^{CW}(X), d_n)$

- thm: X CW complex $\Rightarrow H_n^{CW}(X) \simeq H_n(X)$
- ex: $X = S^n \times S^n$, $n \geq 1$

1 0-cell

1 n-cell

$S^n \times S^n$: 1 0-cell

2 n-cell | $2n$-cell

$$C_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z} \oplus \mathbb{Z} & k = n \\ 0 & \text{else} \end{cases}$$

$\wr\wr$

$H_k(S^n \times S^n)$

b/c $n \geq 1$, all $d_n = 0$

- $\mathbb{C}P^n$: complex lines in $\mathbb{C}^{n+1}$ thru origin $= S^{2n+1}/S^1$

by S^1 action: $w(z_1, \ldots, z_{n+1}) = (wz_1, \ldots, wz_{n+1})$

$\hookrightarrow$ obtained by attaching D_+^{2n} to $\mathbb{C}P^{n-1}$ $\quad D_+^{2n} \sim$

$(n=1)$: $D_+^2 \to \mathbb{C}P^1 = S^3/S^1$, $\partial D_+^2 = S^1 \xrightarrow{\text{onto}} S^1/S^1 = pt$

$\Rightarrow \mathbb{C}P^1 = S^2$

$$H_k(\mathbb{C}P^n) = C_k^{CW}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 0, 2, 4, \ldots n \\ 0 & \text{else} \end{cases}$$

$\frac{F(G)}{G}$

not right @ $n=1$

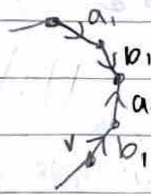
$\partial \Delta^n_\alpha \to X^{n-1} \to X^{n-1}/X^{n-1} - e^{n-1}_\beta = S^{n-1}$

↑ degree of this

$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$, d_1 = simplicial boundary

- ex: $X = \Sigma_g$ (genus g surface)

$4g$ gon

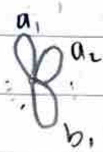


$d_1 = v - v = 0$

$0 \to C_2(X) \xrightarrow{d_2} C_1(X) \to C_0(X) \to 0$

$\quad \mathbb{Z} \qquad \mathbb{Z}^{2g} \qquad \mathbb{Z}$

1 vertex, $2g$ edges, 1 face $\quad X_1 =$



projection

nullhomotopic

$d_2 = 0$

- ex $X = \mathbb{RP}^n = S^n / x \sim -x = D^n_+ / \underset{x \text{ on } \partial D^n_+}{x \sim -x}$

D^n_+ = upper hemisphere

$\partial D^n_+ / x \sim -x = \mathbb{RP}^{n-1} \Rightarrow \mathbb{RP}^n$ obtained by attaching D^n to $\mathbb{RP}^{n-1}$

$C_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & 0 \le k \le n \\ 0 & \text{else} \end{cases}$

$d_n: C_n(X) \to C_{n-1}(X)$

$\quad \mathbb{Z} \qquad \mathbb{Z}$

$d_n(D^+_n) =$ degree of $\left(\partial D^n_+ \xrightarrow{\varphi} \partial D^n_+ / x \sim -x = \mathbb{RP}^{n-1} \xrightarrow{q} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}\right)$

$(n=2)$

q = homeomorphism

$\partial D^n_+ = S^1$

$(z \mapsto z^2)$ $\quad \mathbb{RP}^0 = pt$ $\quad \Rightarrow \deg = 2$

$(n>1)$: $\deg = \underset{\deg(id)}{1} + \underset{\deg(\text{antipodal} \circ id)}{(-1)^n} = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

id

antipodal $\circ$ id

$z \mapsto 2z$

$0 \to C_n(X) \xrightarrow{0/2} \cdots \xrightarrow{2} C_3(X) \xrightarrow{0} C_2(X) \xrightarrow{2} C_1(X) \xrightarrow{0} C_0(X) \to 0$

$\quad \mathbb{Z} \qquad \mathbb{Z} \qquad \mathbb{Z} \qquad \mathbb{Z} \qquad \mathbb{Z}$

$H_0(X) = \mathbb{Z}$, $\underset{0<k<n}{}$ $H_k(X) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ (k odd) $\quad H_k(X) = 0/0 = 0$ (k even)

@ n: $H_n(X) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}$

- def: moore space (simply conn) CW complex w/ homology in a single degree G: $M_n(G_n)$
- $X = \bigvee M_n(G_n) \Rightarrow \tilde{H}_i(X) = \bigoplus_n \tilde{H}_i(M_n(G_n))$

so just need to create a space with homology for a single deg, all else 0

$C = \frac{A}{B} \Rightarrow 0 \to B \to A \to C \to 0$ exact $\quad \operatorname{rank}(A) = \operatorname{rank}(B) + \operatorname{rank}(C)$

- def: let X be finite CW complex, n, $\chi(X) = \sum_{i=0}^{n} (-1)^i (\#\, i\text{-cells})$
- thm: $\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(X))$
  - $H_i(X)$ finitely generated, so $H_i(X) \simeq \mathbb{Z}^r \oplus$ (finite group), $\operatorname{rank}(H_i(X)) = r$
- Cor: $\chi(X)$ is homotopy equivalent invariant
- homology w/ coeffs: let G be abelian group.
  $C_n(X;G) = \left\{ \sum_{\alpha}^{\text{finite}} g_\alpha \sigma_\alpha \mid g_\alpha \in G,\ \sigma_\alpha : \Delta^n \to X \right\}$ $\quad C_n(X,A;G) = C_n(X;G) / C_n(A;G)$
- Lemma: $f: S^n \to S^n$, if $f(x) = -f(x)$ then $\deg(f)$ is odd
  ↳ implies Borsuk Ulam

long homology seq of 2-fold cover $\tilde{X} \xrightarrow{p} X$ (in this lemma, $S^n \to \mathbb{R}P^n$)

$0 \to C_n(X;\mathbb{Z}_2) \xrightarrow{\tau} C_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} C_n(X,\mathbb{Z}_2) \to 0$

↳ transfer / wrong way map ↳ induced map on chains

$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow^{\sigma^+} \nearrow^{\sigma^-} & \downarrow p \\ \Delta^n & \xrightarrow{\sigma} & X \end{array}$

two unique lifts

$\tau(\sigma) := \sigma^+ + \sigma^-$

1) τ is injective
2) $p_\#$ is surjective ($p_\#(\sum_\alpha \sigma_\alpha^+) = \sum \sigma_\alpha \in C_n(X,\mathbb{Z}_2)$)
3) $\operatorname{Im} \tau \subseteq \operatorname{Ker}(p_\#)$: $p_\#(\sum_\alpha \sigma_\alpha^+ + \sigma_\alpha^-)$
   $= \sum_\alpha p\sigma_\alpha^+ + p\sigma_\alpha^- = \sum 2p\sigma_\alpha = 0$ (working in $\mathbb{Z}_2$)
4) $\operatorname{Im} \tau = \operatorname{Ker}(p_\#)$

$\Rightarrow$ short exact seq on chains $\Rightarrow$ long exact seq on homology

$\boxed{\text{working in } \mathbb{Z}_2 \text{ coeffs}}$, suppress for notation:

$0 = H_{n+1}(\mathbb{R}P^n) \xrightarrow{c} \underset{\mathbb{Z}_2}{H_n(\mathbb{R}P^n)} \xrightarrow{\sim\,\operatorname{tr}} \underset{\mathbb{Z}_2}{H_n(S^n)} \xrightarrow{0\;p_*} \underset{\mathbb{Z}_2}{H_n(\mathbb{R}P^n)} \xrightarrow{\sim\,c} \underset{\mathbb{Z}_2}{H_{n-1}(\mathbb{R}P^n)} \xrightarrow{0} \underset{0}{H_{n-1}(S^n)} \Rightarrow \ldots$

$\ldots \to \underset{0}{H_i(S^n)} \xrightarrow{0} \underset{\mathbb{Z}_2}{H_i(\mathbb{R}P^n)} \xrightarrow{\sim} \underset{\mathbb{Z}_2}{H_{i-1}(\mathbb{R}P^n)} \xrightarrow{0} \underset{0}{H_{i-1}(S^n)} \to \ldots$

$\ldots \to \underset{0}{H_1(S^n)} \to \underset{\mathbb{Z}_2}{H_1(\mathbb{R}P^n)} \xrightarrow{\sim} \underset{\mathbb{Z}_2}{H_0(\mathbb{R}P^n)} \xrightarrow{0} \underset{\mathbb{Z}_2}{H_0(S^n)} \xrightarrow{\sim} \underset{\mathbb{Z}_2}{H_0(\mathbb{R}P^n)} \to 0$

Hom functor: takes in two abel. groups A, G, outputs abel. group of all homomorphisms $A \to G$

$(\varphi_1 + \varphi_2)(a) = \varphi_1(a) + \varphi_2(a)$

identity: $0(a) = 0_G$ (zero)

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, dashed, "\varphi f"] & \\ B \arrow[r, "\varphi"'] & G \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(A,G) \\ \mathrm{Hom}(B,G) \arrow[u, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \varphi \circ f = f^*(\varphi) \\ \varphi \arrow[u, maps to] \end{tikzcd}$$

1) $0^* = 0$ $\quad (0^*(\varphi) = \varphi \circ 0 = 0)$

2) f isomorphism $\Rightarrow f^*$ isomorphism

3) f onto $\Rightarrow f^*$ 1-1 $\qquad [f \text{ 1-1} \not\Rightarrow f^* \text{ onto}]$

4) Hom is right exact: $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact

then $\mathrm{Hom}(A,G) \xleftarrow{f^*} \mathrm{Hom}(B,G) \xleftarrow{g^*} \mathrm{Hom}(C,G) \leftarrow 0$ is exact

5) $\mathrm{Hom}(\bigoplus_\alpha A_\alpha, G) = \prod_\alpha \mathrm{Hom}(A_\alpha, G)$

6) $\mathrm{Hom}(\mathbb{Z}, G) \cong G$

7) $\mathrm{Hom}(\mathbb{Z}_m, G) \cong \ker(G \xrightarrow{\cdot m} G)$

8) $(f \circ g)^* = g^* \circ f^*$

} can compute all finitely generated groups

$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \to C_{n-1} \to \ldots$ $\quad$ fix G

$\leftarrow \mathrm{Hom}(C_{n+1}, G) \xleftarrow{\partial_{n+1}^*} \mathrm{Hom}(C_n, G) \xleftarrow{\partial_n^*} \mathrm{Hom}(C_{n-1}, G)$

$\partial_{n+1}^* \partial_n^* = (\partial_{n+1}\partial_n)^* = 0^* = 0$

$$H^n(C, G) = \frac{\ker(\partial_{n+1}^*)}{\mathrm{Im}(\partial_n^*)} = \frac{Z^n}{B^n} = \frac{n\text{-co-cycles}}{n\text{-coboundaries}}$$

there is natural map $H^n(C,G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G)$

lemma: h is split surjective: $\exists s$ homomorphism

s.t. $hs = id$ $\qquad H^n(C,G) \xleftarrow{s} \mathrm{Hom}(H_n(C), G)$

$\Rightarrow$ so h is surjective $\qquad \mathrm{Hom} \xrightarrow{s} H^n \xrightarrow{h} \mathrm{Hom}$

injective $\qquad$ surjective

not always injective

- Thm: for any 2 abelian A, G $\exists$ abelian group $\mathrm{Ext}(A,G)$, natural in A, G s.t.

a) $\mathrm{Ext}(\bigoplus A_\alpha, G) = \bigoplus_\alpha \mathrm{Ext}(A_\alpha, G)$ $\qquad$ * $\mathrm{Ext}(\text{free}, G) = 0$

b) $\mathrm{Ext}(\mathbb{Z}, G) = 0$ $\qquad$ c) $\mathrm{Ext}(\mathbb{Z}_m, G) \cong G/mG$

- Thm (Universal Coeffs): $\exists$ natural in C short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

s

- Corollary: If a chain map $\varphi: (C, \partial) \to (C', \partial')$ induces $\cong$ on homology, then it induces $\cong$ on cohomology (first lemma on previous)
- Cor: Suppose (C, ∂) is a chain complex and $H_n(C)$, $H_{n-1}(C)$ are finitely gen. with torsion subgroups T_n, T_{n-1} (torsion = finite order elts).
  Then $H^n(C; \mathbb{Z}) \cong (H_n(C)/T_n) \oplus T_{n-1}$
  $\to$ changing torsion T_n doesn't affect cohomology (but it sees T_{n-1})
  $\to H^n(C;\mathbb{Z}) \cong \mathrm{Hom}(H_n(C), G) \oplus \mathrm{Ext}(H_{n-1}(C), \mathbb{Z})$
- def: X be a space (CW complex, simplicial)
  $H^n(C; G)$ is the homology of
  $$\dots \leftarrow \mathrm{Hom}(C_n(X), G) \xleftarrow{\partial_n^*} \mathrm{Hom}(C_{n-1}(X), G) \leftarrow \dots$$
  $$\partial_n^* = \delta_n$$
- ex($n=0$): $H_{n-1}(X) = H_{-1}(X) = 0$ so by Univ. Coeff
  $$H^0(X, G) \cong \mathrm{Hom}(H_0(X), G) = \mathrm{Hom}(\oplus_\alpha \underbrace{H_0(X_\alpha)}_{\mathbb{Z}}, G) \cong \prod_\alpha \mathrm{Hom}(\mathbb{Z}, G) \cong \prod_\alpha G$$
  $\downarrow$
  functions $X \to G$ $\quad \varphi \to \varphi|_X$ $\quad$ (X is a basis for $C_0(X)$)
  φ is cocycle $\iff$ φ is constant on path components
- ex: ($n=1$): $\mathrm{Ext}(H_0(X), G) = 0$ (b/c $H_0(X)$ is free abelian)
  $\Rightarrow H^1(X; G) = \mathrm{Hom}(H_1(X), G)$
  Say G has no finite order elts, $H_1(X) \cong \mathbb{Z}^r \oplus \underset{\text{finite}}{T_1}$
  Then $H^1(X; G) = \mathrm{Hom}(\mathbb{Z}^r \oplus T_1, G) \cong G^r \oplus \underbrace{\mathrm{Hom}(T_1, G)}_{0} \cong G^r$
- ex ($n=0$): $\tilde{H}^0(X, G) \cong \dfrac{\text{functions } X \to G \text{ constant on path comps}}{\text{constant functs } X \to G}$

Goal: product structure on $\oplus_{n \geq 0} H^n(X, R)$ $\quad R$: comm. ring w/ unit $(\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}_n)$
- def: cup product of cochains $\varphi \in C^k(X; R)$, $\psi \in C^l(X; R)$
  $\varphi \cup \psi \in C^{k+l}(X; R)$, fix $\sigma: \Delta^{k+l} \to X$ (basis elt)
  $$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0 \dots v_k]}) \cdot \psi(\sigma|_{[v_k \dots v_{k+l}]})$$
  ↑ R-linear product $\qquad$ mult in R
- lemma: $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

cohomology ring

$$\bigoplus_{n\geq 0} H^n(X;R) =: H^*(X;R)$$

Cor: a) If $\delta\varphi = 0 = \delta\psi$ then $\delta(\varphi \cup \psi) = 0$ (cocycle · cocycle = cocycle)
b) product of cocycle and coboundary is coboundary

Properties: 1) If $f: X \to Y$ cont, $f^*: H^*(Y;R) \to H^*(X;R)$
$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$

2) cup prod is associative on cochains

3) $\alpha \in H^k(X;R)$, $\beta \in H^l(X;R) \Rightarrow \alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$

- ex: any map $S^2 \to X_g$ is trivial on 2nd cohomology
  - lift to univ cover $\mathbb{R}^2$, nullhomotopic

0 map
$H^1(X_g) \to H^1(S^2) = \mathrm{Hom}(\underbrace{H_1(S^2)}_{0}, \mathbb{Z}) = 0$

$H^2(X_g) \xrightarrow{f^*} H^2(S^2) = \mathbb{Z}$

generator $\gamma = \alpha_1 \cup \beta_1$ $\quad f^*\gamma = f^*\alpha_1 \cup f^*\beta_1 = 0 \cdot 0 = 0 \Rightarrow$ so $f^* = 0$

- ex: any map $\underset{S^1\times S^1}{X_1} \to X_2$ is trivial on 2nd cohomology

$H^1(X_2) = \overset{\alpha_1}{\mathbb{Z}} \oplus \overset{\alpha_2}{\mathbb{Z}} \oplus \overset{\beta_1}{\mathbb{Z}} \oplus \overset{\beta_2}{\mathbb{Z}}$ $\qquad H^1(X_1) = \overset{\tau_1}{\mathbb{Z}} \oplus \overset{\tau_2}{\mathbb{Z}}$

$H^2(X_2) = \underset{\substack{\alpha_1 \cup \beta_1 \\ = \alpha_2 \cup \beta_2}}{\mathbb{Z}}$

$f^*(a_i) = a_{1i}\tau_1 + a_{2i}\tau_2$

$f^*(b_i) = b_{1i}\tau_1 + b_{2i}\tau_2$ $\qquad \tau_i \cup \tau_i = -\tau_i\tau_i \Rightarrow \tau_i \cup \tau_i = 0$

$f^*(a_i) \cup f^*(b_i) = (a_{1i}\tau_1 + a_{2i}\tau_2)(b_{1i}\tau_1 + b_{2i}\tau_2)$

$= (a_{1i}b_{2i} - a_{2i}b_{1i})\tau_1\tau_2 \quad (\tau_i^2 = 0,\ \tau_1 \cup \tau_2 = -\tau_2 \cup \tau_1)$

sps $\neq 0$. Then $= \det \begin{bmatrix} a_{1i} & b_{1i} \\ a_{2i} & b_{2i} \end{bmatrix} \neq 0$, so col vecs lin. indep

[...] they're dependent ∴

- thm (Poincaré duality): let M be an n-manifold, closed (compact w/o boundary), then $H^n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and for each k, the bilinear map

$$H^k(M;\mathbb{Z}_2) \times H^{n-k}(M;\mathbb{Z}_2) \to H^n(M;\mathbb{Z}_2)$$
$$\alpha \qquad \beta \qquad \to \alpha \cup \beta$$

has the property that $\forall \alpha$, the map

$$H^k(M;\mathbb{Z}_2) \to \mathrm{Hom}(H^{n-k}(M;\mathbb{Z}_2), \mathbb{Z}_2) \quad \text{is an isomorphism}$$
$$\alpha \to (\beta \to \alpha \cup \beta)$$

- Ex: Cohomology ring $H^*(\mathbb{RP}^n; \mathbb{Z}_2)$ is $\mathbb{Z}_2[x]/x^{n+1}$

Pf: the inclusion $\mathbb{RP}^k \hookrightarrow \mathbb{RP}^n$ is an isomorphism on $H^{i \le k}(\cdot\,; \mathbb{Z}_2)$

So $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ is an H^1 isomorphism, let x be the generator.

Induction on n, $n=1$ ($\mathbb{RP}^1 = S^1$) ✓

Induction step: Poincaré duality, isomorphism shows nonzero map.

- M n-manifold: local orientation of M @ $x \in M$ is a choice of generator $\mu_x \in$ for $H_n(M, M-x)$ $\simeq$ (excision, $x \in D^n$) $H_n(D^n, D^n - x) \simeq$ (LES) $H_{n-1}(D^n - x) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$
- def: an orientation is a function $x \mapsto \mu_x$ that is locally consistent: if any $x \in M$ has a disk D^n neighborhood st $\forall y \in D^n$,

$$H_n(M, M - D^n) \xrightarrow{\sim} H_n(M, M-y) \ni \mu_y$$
$$\downarrow \wr \text{ excision}$$
$$\mu_x \in H_n(M, M-x)$$

$\mu_x \mapsto \mu_y$

generally:

- Fact: any connected n-manifold M has an orientable 2-fold covering space (two choices for generator at each pt) $\tilde{M} = \{\mu_x \mid x \in M\} \to M$
  - ↳ if M is already orientable, $\tilde{M} = M \sqcup M$
- Cor: If M is connected and $\pi_1(M)$ has no index 2 subgroup (no connected 2-fold cover) then M is orientable
- Fact: If R is comm ring w/ unit, $H_n(M; M-x; R) \simeq R$ local orientation μ_x, st. $R\mu_x = R$ ($r\mu_x = s$ is always solvable)
- any n-manifold is $\mathbb{Z}_2$ orientable (choice of generator must be 1)
- def: a class $[M]$ in $H_n(M; R)$ (M R oriented n-manifold) is the R-fundamental class if the inclusion $M \to (M, M-x)$ sends $[M] \to \mu_x$
  - ↳ $M = \mathbb{R}^n$, no fundamental class
- thm: If M is closed (compact, no boundary) and R-oriented, then $H_n(M; R) \simeq R$ generated by $[M] \in H_n(M;R)$
- thm (Poincaré Duality): If M closed n-manifold R oriented, w/ $[M]$ fundamental class then $H^k(M; R) \underset{D}{\xrightarrow{\sim}} H_{n-k}(M; R)$ $D(\beta) = [M] \cap \beta$
  - ↳ so $H^k(M; R) \simeq 0$ for $k > n$, H_k $k > n$
  - ↳ if M connected, $H^n(M; R) \simeq H_0(M; R) \simeq R$

$H^0(M; R) \simeq H_n(M; R) \simeq R$

$$\bigoplus_{n\geq 0} H^n(X;R) =: H^*(X;R)$$ ← cohomology ring

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- ex: any map $S^2 \to X_g$ is trivial on 2^{nd} cohomology
  - lift to univ cover $\mathbb{R}^2$, nullhomotopic

0 map: $H^1(X_g) \to H^1(S^2) = \mathrm{Hom}(\underbrace{H_1(S^2)}_{0}, \mathbb{Z}) = 0$

generator: $H^2(X_g) \xrightarrow{f^*} H^2(S^2) = \mathbb{Z}$

$\gamma = \alpha_1 \cup \beta_1 \quad f^*\gamma = f^*\alpha_1 \cup f^*\beta_1 = 0 \cdot 0 = 0 \Rightarrow$ so $f^* = 0$

- ex: any map $\underset{\text{S}^1\times\text{S}^1}{X_1} \to X_2$ is trivial on 2^{nd} cohomology

$H^1(X_2) = \overset{\alpha_1}{\mathbb{Z}} \oplus \overset{\alpha_2}{\mathbb{Z}} \oplus \overset{\beta_1}{\mathbb{Z}} \oplus \overset{\beta_2}{\mathbb{Z}}$ $\qquad H^1(X_1) = \overset{\tau_1}{\mathbb{Z}} \oplus \overset{\tau_2}{\mathbb{Z}}$

$H^2(X_2) = \underset{\substack{\alpha_1\cup\beta_1\\=\alpha_2\cup\beta_2}}{\mathbb{Z}}$

$f^*(a_i) = a_{1i}\tau_1 + a_{2i}\tau_2$

$f^*(b_i) = b_{1i}\tau_1 + b_{2i}\tau_2$ $\qquad \tau_i \cup \tau_i = -\tau_i \tau_i$

$f^*(a_i) \cup f^*(b_i) = (a_{1i}\tau_1 + a_{2i}\tau_2)(b_{1i}\tau_1 + b_{2i}\tau_2)$ $\qquad \Rightarrow \tau_i \cup \tau_i = 0$

$= (a_{1i}b_{2i} - a_{2i}b_{1i})\tau_1\tau_2 \qquad (\tau_i^2 = 0,\ \tau_1 \cup \tau_2 = -\tau_2 \cup \tau_1)$

sps $\neq 0$. Then $= \det \begin{bmatrix} a_{1i} & b_{1i} \\ a_{2i} & b_{2i} \end{bmatrix} \neq 0$, so col vecs lin. indep

[...] they are dependent :(

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$$H^k(M;\mathbb{Z}_2) \times H^{n-k}(M;\mathbb{Z}_2) \to H^n(M;\mathbb{Z}_2)$$
$$\alpha \qquad \beta \quad \to \quad \alpha \cup \beta$$

has the property that $\forall \alpha$, the map

$$H^k(M;\mathbb{Z}_2) \to \mathrm{Hom}(H^{n-k}(M;\mathbb{Z}_2), \mathbb{Z}_2)$$ is an isomorphism
$$\alpha \to (\beta \to \alpha \cup \beta)$$

- Ex: Cohomology ring $H^*(\mathbb{RP}^n;\mathbb{Z}_2)$ is $\mathbb{Z}_2[x]/x^{n+1}$
Pf: the inclusion $\mathbb{RP}^k \hookrightarrow \mathbb{RP}^n$ is an isomorphism on $H^{i\le k}(\cdot;\mathbb{Z}_2)$
So $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ is an H^1 isomorphism, let x be the generator.
Induction on n, $n=1$ ($\mathbb{RP}^1 = S^1$) ✓
Inductive step: Poincaré duality, isomorphism shows nonzero map.
- M n-manifold: local orientation of M @ $x\in M$ is a choice of generator $\mu_x \in$ for $H_n(M, M-x)$
$\simeq$ (excision, $x\in D^n$) $H^n(D^n, D^n - x) \simeq$ (LES) $H_{n-1}(D^n - x) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$
- def: an orientation is a function $x \mapsto \mu_x$ that is locally consistent:
if every $x\in M$ has a disk D^n neighborhood st $\forall y \in D^n$, $\mu_x \leftrightarrow \mu_y$

$$\begin{array}{ccc} H_n(M, M-D^n) & \xrightarrow{\sim} & H_n(M, M-y) \ni \mu_y \\ \downarrow \wr \text{ excision} & & \\ \mu_x \in H_n(M, M-x) & & \end{array}$$

generally:
- Fact: any connected n-manifold M has an orientable 2-fold covering space
(two choices for generator at each pt) $\tilde{M} = \{\mu_x \mid x \in M\} \to M$
↳ if M is already orientable, $\tilde{M} = M \sqcup M$
- Cor: If M is connected and $\pi_1(M)$ has no index 2 subgroup
(no connected 2-fold cover) then M is orientable
- Fact: If R is comm ring w/ unit, $H_n(M; M-x; R) \simeq R$
local orientation μ_x, st. $R\mu_x = R$ ($r\mu_x = s$ is always solvable)
- any n-manifold is $\mathbb{Z}_2$ orientable (choice of generator must be 1)
- def: a class $[M]$ in $H_n(M;R)$ (M R oriented n-manifold) is
the R-fundamental class if the inclusion $M \to (M, M-x)$
sends $[M] \to \mu_x$
↳ $M = \mathbb{R}^n$, no fundamental class
- thm: If M is closed (compact, no boundary) and R-oriented,
then $H_n(M;R) \simeq R$ generated by $[M] \in H_n(M;R)$
- thm (Poincaré Duality): If M closed n-manifold R oriented, w/ $[M]$ fundamental class
then $H^k(M;R) \underset{D}{\xrightarrow{\sim}} H_{n-k}(M;R)$ $D(\beta) = [M] \frown \beta$
↳ so $H^k(M;R) \simeq 0$ for $k>n$
↳ if M connected, $H^n(M;R) \simeq H_0(M;R) \simeq R$
$H^0(M;R) \simeq H_n(M;R) \simeq R$

def: $\text{rank}(H_i(M)) = b_i(M)$ Betti number

- ex: $H^1(M;\mathbb{Z}) \cong \text{Hom}(H_1(M);\mathbb{Z}) \cong \text{Hom}(\mathbb{Z}^r \oplus \text{finite}, \mathbb{Z}) \cong \mathbb{Z}^r$
  (universal coeffs; $\wr\wr$ ← if M $\mathbb{Z}$-orientable, P.D.; finitely generated if M CW complex (assume))
  $\Rightarrow H_{n-1}(M)$ torsion free
- ex: let M n-manifold closed $\mathbb{Z}$-orientable, homeo to finite CW complex
  Claim: If $\dim M = n$ odd, $\chi(M) = 0$
  $\chi(M) = \sum_{i=0}^{n} (-1)^i (\#\, i\text{-cells}) = \sum_{i=0}^{n} (-1)^i \text{rank}(H_i(M)) = b_0 - b_1 + b_2 \ldots + b_n$ ...
  $H_i(M) \underset{PD}{\cong} H^{n-i}(M) \underset{U.C.}{=} \text{Hom}(H_{n-i}(M,\mathbb{Z})) = \mathbb{Z}^{b_{n-i}}$ (finitely gen)
  $\Rightarrow b_i = b_{n-i} \Rightarrow \chi(M) = 0$.
- If M not $\mathbb{Z}$ orientable, $\tilde{M}$ is orientable. any cell in M lifts (simply conn) to 2 cells of same dim. $\#\, i$ cells in $\tilde{M} = 2\#\, i$ cells in M
  $\Rightarrow \chi(\tilde{M}) = 2\chi(M)$, and $\chi(\tilde{M}) = 0$ by above.
- def: Cap product: $\cap : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X,R)$ $(k \geq \ell)$
  $\sigma : [v_0 \ldots v_k] \to X$, $\sigma \cap \varphi = \sigma|_{[v_\ell \ldots v_k]} \underset{R}{\cdot} \underbrace{\varphi(\sigma|_{[v_0,\ldots v_\ell]})}_{\text{scaling in } R}$
  ↳ gives $\cap$ product on homology
- Fact: $(\varphi \cup \psi)(\varepsilon) = \psi(\varepsilon \cap \varphi)$
- Poincaré duality isomorphism $H^k(M;R) \cong H_{n-k}(M;R)$ $\forall k$
  is given by $\beta \to [M] \cap \beta =: D(\beta)$ (duality)

- ex: $\mathbb{C}P^2 \underset{g}{\overset{f}{\rightleftarrows}} S^2 \times S^2$, can f, g be $\cong$ on 4th homology? (No)
  Compute H^* of both } cup product involved

$R = \mathbb{Z}_2$ ex: $\underset{\alpha \in}{H^{n-k}(M;\mathbb{Z}_2)} \overset{U.C.}{\cong} \text{Hom}_{\mathbb{Z}_2}(H_{n-k}(M), \mathbb{Z}_2)$ (no Ext b/c $\mathbb{Z}_2$)
(poincaré duality) $\to D^* \text{Hom}_{\mathbb{Z}_2}(H^k(M), \mathbb{Z}_2)$
α is a hom. $H^k(M) \to \mathbb{Z}_2$
so $\exists$ $\mathbb{Z}_2$-bilinear nonsingular map: $H^{n-k}(M;\mathbb{Z}_2) \times H^k(M;\mathbb{Z}_2) \to \mathbb{Z}_2$
$\alpha \quad \beta \mapsto D^* h(\alpha)\beta$
$D^* h(\alpha)\beta \underset{\text{duality}}{=} h(\alpha) D(\beta) \underset{\text{def of } D}{=} h(\alpha)([M] \cap \beta) = (\beta \cup \alpha)[M]$
$(\varphi \cup \psi)(\varepsilon) = \psi(\varepsilon \cap \varphi)$

Ext $= 0$ b/c free grp.

- ex: $S^n \times S^m$, $R = \mathbb{Z}$

$H^k(S^n \times S^m) = \mathrm{Hom}(H_k(S^n \times S^m), \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} & k = m+n \\ \mathbb{Z} & k = m, \text{ or } n \end{cases}$ $m \neq n$

let $x_n \in H^n(X)$, $x_m \in H^m(X)$ be generators,

$x_n \cup x_m \in H^{m+n}(X)$ $= \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} & k=n+n \\ \mathbb{Z} \oplus \mathbb{Z} & k=m=n \end{cases}$ $m=n$

is this a generator?

nonsingular form $\Rightarrow x_n \cup \left(\text{some class in } \underset{\substack{\parallel \\ \mathbb{Z} \text{ gen by } x_m}}{H^m(X)}\right) = 1 \in H^{n+m}(X)$.

if (some) is $2x_m$ then $x_n \cup 2x_m = 2(x_n \cup x_m)$ is even

so (some) must be x_m or $-x_m$

$(m \neq n)$: $x_n \cup x_n = 0$ (no cohomology in $H^{n+n}(X)$)

$x_m \cup x_m = 0$

$(m = n)$: $x_n^2 = \pm x_n \cup x_m = 1$ (generator for $n+m = n+n$)

Cohomology w/ compact support

(simplicial) let X be locally finite Δ-complex (every compact set in X only meets finitely many simplices)

- def: $\Delta^k_c(X) = \{ \varphi \in \Delta^k(X) = \mathrm{Hom}(\Delta_k(X), \mathbb{Z}) : \varphi \text{ is nonzero on finitely many } k\text{-simplices} \}$

simplicial k-cochains w/ compact support

$\boxed{\text{cochain}}$ $H^k_c(X)$ = homology of this chain complex

- thm (PD): If X is R-oriented n-manifold, that is a Δ-complex, then $H_c^k(X) \cong H_{n-k}(X)$
- ex ($k = n$): $H^n_c(X) \cong H_0(X) = \mathbb{Z}$
- ex: $X = \mathbb{R}$

goal

$0 \neq H^1_c(X, \mathbb{Z}) = \Delta^1_c(X)/\text{coboundary}$

① b/c no 2-simplices

$S : \Delta^1_c(X) \to \mathbb{Z}$

$\varphi \mapsto \sum \varphi([i, i+1])$

finite sum

$S(\delta\psi) = \sum \delta\psi([i, i+1]) = 0$

$= \psi(\partial[i, i+1])$

$= \psi(i+1) - \psi(i)$

② so $\Delta^1_c(X) \xrightarrow{S} \mathbb{Z}$

$\searrow H^1_c(X) \nearrow$ descends to cohomology

S is onto $\Rightarrow H^1_c(X)$ is nontrivial