Hyperbolic Knots

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Abstract

In this survey, we explore the connections between knot theory and the geometry of hyperbolic 3-space. In particular, we investigate a knot/link invariant, the hyperbolic volume of the knot complement. We provide a full computation of this invariant for the figure-eight knot.

1 Introduction

The discovery of the relation between hyperbolic structure and knot complements was independently discovered by Robert Riley and William Thurston in the mid-1970s [Ril13]. Riley was able to put a hyperbolic structure on the figure-8 knot complement by taking a quotient of $\mathbb{H}^3$ by a group relating to the figure-8 knot. In contrast to Riley's algebraic approach, Thurston's approach was geometric; he directly decomposed the figure-8 knot complement into two ideal tetrahedra, and this is the approach we will follow in this paper. This connection between topology and geometry was surprising - at the time, topologists saw hyperbolic geometry as an "arcane side branch of mathematics" [Thu94]. This relation was explored further with Thurston's famous Geometrization Conjecture, though it took some time for topologists to "understand what [it] meant, what it was good for, and why it was relevant" [Thu94].

2 Preliminaries

We first give some definitions to remind the reader of the essentials from hyperbolic geometry and knot theory relevant to this paper.

2.1 Hyperbolic Geometry

Definition 2.1. We define *hyperbolic 3-space* as $\mathbb{H}^3 = \{(x, y, u) \in \mathbb{R}^3 : u > 0\}$. The *hyperbolic length* of a piecewise differentiable curve $\gamma\colon [a, b] \to \mathbb{H}^3$ with parameterization $\gamma(t) = (x(t), y(t), u(t))$ is $l_{\text{hyp}}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2+y'(t)^2+u'(t)^2}}{u(t)} dt$. The *hyperbolic distance* between two points $P, Q \in \mathbb{H}^3$ is defined $d_{\text{hyp}}(P, Q) = \inf\{l_{\text{hyp}}(\gamma)\}$, where the infimum is taken over all piecewise differentiable curves γ from P to Q.

There is another model for hyperbolic geometry, namely, the Poincaré ball model. One can intuitively imagine this by taking the upper half space and dragging/collapsing the points at infinity upwards, to obtain a ball B^3 (note the points at infinity, or equivalently, the boundary of the ball is not included as points in $\mathbb{H}^3$). We will swap between these models, as each model has its advantages and disadvantages that we can leverage. More details on hyperbolic 3-space can be found in [Bon09].

2.2 Knot Theory

Definition 2.2. A *knot* $K \subseteq S^3$ is a smooth embedding of the circle S^1 in S^3. We often refer to the knot as the image of the embedding.

By referring to the knot as the image of the embedding, we no longer care about the orientation of the embedding (i.e., the knot is a set rather than an embedding). Considering oriented knots are interesting in their own right, but we will not make that distinction here.

As another note, knots can usually be studied in $\mathbb{R}^3$ and S^3 without much meaningful difference. However, if we want to use geometry to study knots (which we will do later), we want to work in S^3, as we can place a complete hyperbolic metric of finite volume on some manifolds that are subsets of S^3. In $\mathbb{R}^3$, we could not do this. Generally, the fact that S^3 is compact and not contractible is helpful as well.

Definition 2.3. A *link* $L \subseteq S^3$ of k-components is a smooth embedding of a disjoint union of k circles in S^3.

In other words, a link is a finite union of disjoint knots.

Definition 2.4. Two knots K_1, K_2 are *equivalent* if there is an orientation preserving piecewise-linear homeomorphism $h\colon S^3 \to S^3$ such that $h(K_1) = K_2$.

Informally, knots are equivalent if we can deform one without ripping or crossing themselves so that one matches up with the other. This is equivalent to saying two knots K_1, K_2 are equivalent if there is an ambient isotopy between the embeddings. One of the central questions of knot theory is classifying knots up to isotopy.

Definition 2.5. A *knot* (or *link*) *invariant* is a function from the set of knots (or links) to some other set, where equivalent knots must have the same output.

The codomain can be anything; some examples include $\{0, 1\}$ (true/false), $\mathbb{Z}$, $\mathbb{R}$, and the set of polynomials. Invariants are central to the classification of knots − if two knots have different invariant values, then they cannot be equivalent.

Definition 2.6. For a knot K, the knot complement is the open manifold $S^3 \setminus K$.

We now present Thurston's Geometrization Theorem for Knot Complements.

Theorem 2.1 (Thurston, 1974)**.** *For any knot $K \subseteq S^3$, exactly one of the following holds.*

(a) K is a torus knot $T(p, q)$ with $q \geq 2$;

(b) K is a satellite of a nontrivial knot;

(c) $S^3 \setminus K$ (the knot complement) admits a metric d which is

- *(i) complete*
- *(ii) topologically equivalent to the Euclidean metric*
- *(iii) locally isometric to the hyperbolic metric d_{hyp} of $\mathbb{H}^3$.*

Knots which satisfy (c) are said to be *hyperbolic*. For more information on the first two types of knots, see [Bon09] or [Ada84].

3 Hyperbolic Volume

In this section, we'll define the hyperbolic volume and explore how to compute it. Informally, we want to divide the space into hyperbolic tetrahedra, compute the volume of each hyperbolic tetrahedron, and then add the volumes together.

3.1 Hyperbolic Triangulations

Given a manifold, how can we describe it? One method is to triangulate our manifold, which is also a useful technique for computations on manifolds.

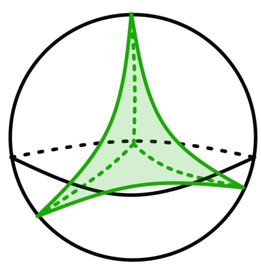
Definition 3.1. A manifold M is a *hyperbolic manifold*, if it is a Riemannian manifold and is locally isometric to $\mathbb{H}^n$.

That is to say, each point of M has a neighborhood which is isometric to a neighborhood of $\mathbb{H}^n$. We will only care about when $n = 3$. Our goal is now to build a hyperbolic manifold, and we will do so by gluing together hyperbolic polyhedra via hyperbolic isometries.

Definition 3.2. A *hyperbolic polyhedron* is a compact subset of $\mathbb{H}^3$ obtained by taking the intersection of a finite collection of half-spaces. An *ideal polyhedron* is a polyhedron with vertices "at infinity."

In the ball model, one can think of an ideal polyhedron as a polyhedron with vertices on the boundary of the ball (that is, at ∞). Now how can we decompose a hyperbolic manifold into (ideal) polyhedra? We will do a concrete example in Section 5, but we leave the more general case to the discussion in [Thu79] and [Pur20]. If we can decompose our manifold into tetrahedra only, we say our manifold has been *triangulated*.

Figure 1: An ideal tetrahedra in the ball model.



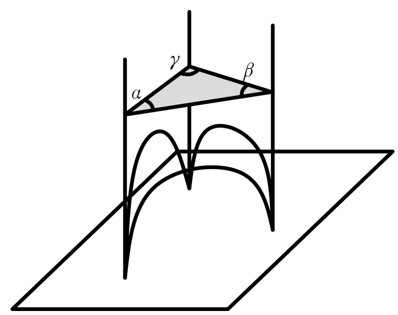
Let's suppose we have a collection of ideal hyperbolic polyhedra and some gluing information. The gluing may or may not give a hyperbolic manifold (in fact, it might not even give a manifold!), but the following theorem gives us a criterion as to when the gluing gives a hyperbolic manifold.

Theorem 3.1. *Let M be obtained by gluing hyperbolic polyhedra (possibly ideal) $P_1, \ldots, P_m$ by identifying their faces in pairs via isometries. Let $P = P_1 \sqcup \ldots \sqcup P_m$ be the disjoint union. Let $q\colon P \to M$ be the quotient map for the gluing. Suppose each point $x \in M$ has a neighborhood U_x and an open mapping $\phi_x\colon U_x \to B_\epsilon(0)$ which is a homeomorphism and restricts to an isometry on each component of $U_x \cap q(P \setminus \partial P)$. Then M inherits the hyperbolic structure.*

We refer to [Lac00] for the proof of this theorem. Informally, this theorem is just saying if every point of M has a neighborhood isometric to a ball in $\mathbb{H}^3$, then M inherits the hyperbolic structure. In the special case of $\mathbb{H}^3$, it *nearly* suffices to check that for every edge, taking the sum of the dihedral angles of all edges glued to this edge, sums to 2π. There is a second condition, but the background to state it is beyond the level of this paper. For further details, see the Gluing Consistency Conditions in [Thu79] or Gluing Equations in [Pur20].

Lemma 3.2. *Let F_1, F_2, F_3 be three faces of an ideal tetrahedron in $\mathbb{H}^3$. Let α, β, γ be the dihedral angles between F_1, F_2, then F_2, F_3, and F_3, F_1, respectively. Then $\alpha + \beta + \gamma = \pi$.*

Proof. Let's work in the upper half space model. By isometries of $\mathbb{H}^3$, we may assume that the hyperplanes that contain F_1, F_2, F_3 are vertical Euclidean planes, since vertical lines are geodesics in $\mathbb{H}^3$. But then α, β, γ are the interior angles of a Euclidean triangle, hence sum to π. □



We call the angles α, β, γ the *dihedral angles* of the tetrahedron.

Lemma 3.3. *Every ideal tetrahedron in $\mathbb{H}^3$ is in direct correspondence (up to orientation-preserving isometries) with triples $\alpha, \beta, \gamma \in (0, \pi)$ whose sum is π. These three angles are the dihedral angles of the tetrahedron.*

Proof idea. Given an ideal tetrahedron, we can get the three angles as described in the proof of the previous lemma. Now given a triple $\alpha, \beta, \gamma \in (0, \pi)$ summing to π, we can find a Euclidean triangle in the x-y plane (i.e., in $\partial\mathbb{H}^3$) with these angles, and consider the ideal tetrahedron with vertices at the vertices of this triangle, and the point at ∞. We can use hyperbolic isometries to match up any two triangles with the same interior angles. □

For a more in-depth proof, see [Lac00].

3.2 Computing Hyperbolic Volume

We are now ready to compute the volume.

Definition 3.3. The *Lobachevsky function* $\Lambda(\theta) \colon \mathbb{R} \to \mathbb{R}$ is defined

$$\Lambda(\theta) = -\int_0^\theta \ln|2\sin t|dt$$

Theorem 3.4 (Volume of hyperbolic ideal tetrahedra)**.** *Let $\alpha, \beta, \gamma \in (0, \pi)$ such that $\alpha + \beta + \gamma = \pi$, so they form a hyperbolic ideal tetrahedron T. Then the volume is $vol(T) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$.*

A proof of this theorem can be found in [Thu79]. To numerically compute the Lobachevsky function, we can use the following series expansion, which converges relatively quickly.

Lemma 3.5. *The Lobachevsky function has series expansion*

$$\Lambda(\theta) = \theta\left(1 - \ln|2\theta| + \sum_{n=1}^{\infty} \frac{B_n}{2n} \frac{(2\theta)^{2n}}{(2n+1)!}\right)$$

where B_n denotes the even Bernoulli numbers (this may be denoted B_{2n} in some literature).

Note that the derivatives of the Lobachevsky function are

$$\frac{d}{d\theta}\Lambda(\theta) = -\ln|2\sin\theta|$$
$$\frac{d^2}{d\theta^2}\Lambda(\theta) = -\cot\theta$$

Also, it is well known that the Laurent series for $\cot x$ is

$$\cot x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_n}{(2n)!} x^{2n-1} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_n}{(2n)!} x^{2n-1}$$

and so we can integrate this series twice to obtain the desired series expansion. While the series for $\cot x$ only converges for $0 < |x| < \pi$, it can be shown that $\Lambda(\theta)$ is periodic with period π and that the series expansion converges at $\theta = \pi$, so this expansion is valid everywhere.

3.3 Hyperbolic Volume as an Invariant

Definition 3.4 (Hyperbolic volume)**.** Let $M = S^3 \setminus K$ and suppose M can be decomposed into tetrahedra $T_1, \dots T_m$. The *hyperbolic volume*, denoted hypvol, is defined as the sum $\text{hypvol}(S^3 - K) = \sum_{n=1}^{m} \text{vol}(T_n)$. By abuse of terminology, we will say 'the hyperbolic volume of a knot K' to mean the hyperbolic volume of the knot complement $S^3 \setminus K$.

It is not at all obvious that two seemingly different triangulations of the space will have the same volume, nor if the triangulation remains the same if the knot is 'wiggled' a little, meaning this definition may not even be well-defined! However, the following theorem ensures there is only one unique, complete, hyperbolic structure (if a manifold can admit one), up to isometry.

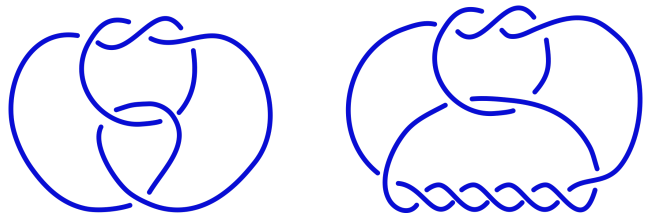
Theorem 3.6 (Mostow's Rigidity Theorem)**.** *Let M and N be complete hyperbolic 3-manifolds with finite volume. If $f\colon M \to N$ is a homotopy equivalence, then f is homotopic to an isometry from M to N.*

For those who do not know what a homotopy equivalence is, a weaker form of the theorem states if M and N are homeomorphic, then the spaces are isometric. A proof of the theorem can be found in [Bon09]. This theorem shows that hyperbolic volume is indeed an invariant, since if we can give a complete hyperbolic structure on a knot complement, then any other complete hyperbolic structure on the space will be isometric, and thus have the same volume! However, this invariant is not a complete classifying invariant: there exist different hyperbolic knots with the same hyperbolic volume.

We finish this section with some fun facts about hyperbolic volume (without proof).

- Figure 2 shows two knots with the same hyperbolic volume, of approximately 2.828122088.
- Only a finite number of knots can have the same hyperbolic volume.
- Hyperbolic volume distinguishes all hyperbolic knots with less than 10 crossings. The values can be found on the database KnotInfo.
- The program SnapPy is used to compute the hyperbolic volume of knots.

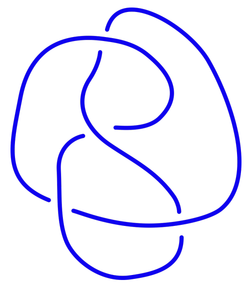
Figure 2: The knots 5_2 and $12n_{242}$.



4 The Figure-Eight Knot

We will analyze the figure-eight knot, the simplest hyperbolic knot. For this section, let K denote the figure-eight knot. Figures in this section were created by myself using Inkscape (a free and open source vector graphics editor, which I highly recommend!), and figures were inspired from [Fra87], [Pur20], and [Seg14].

Figure 3: The figure-eight knot.

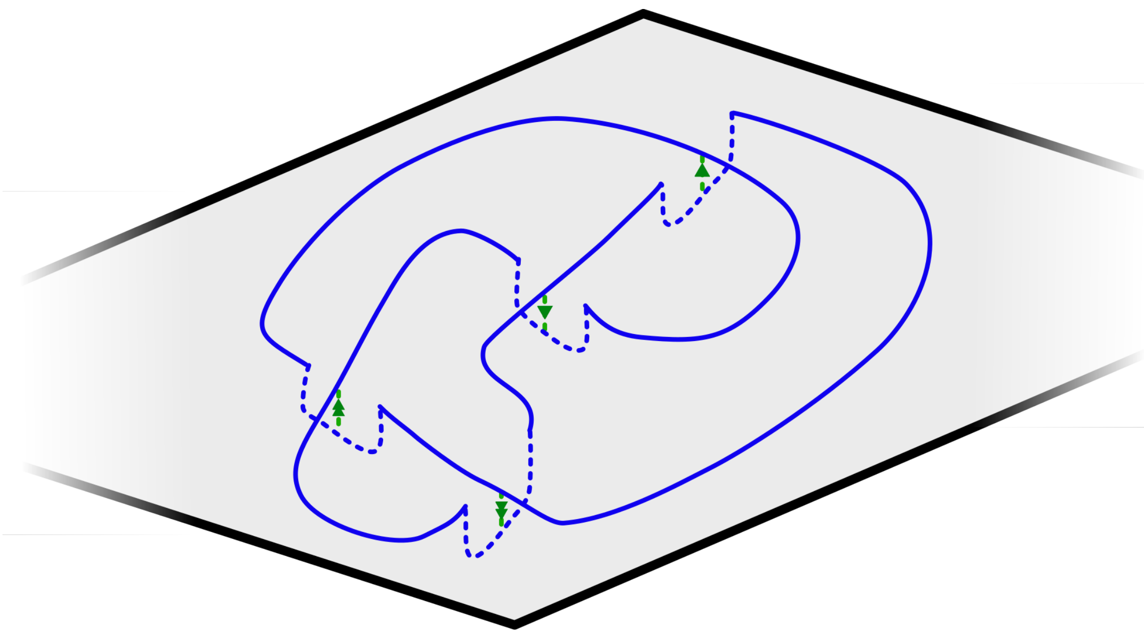


Theorem 4.1 (Riley '73, Thurston '78)**.** *The figure-eight knot complement is a hyperbolic manifold.*

To prove this theorem, our general strategy will be to construct a triangulation of the knot complement, and then show that it admits a hyperbolic structure. Surprisingly, we will only need two tetrahedra!

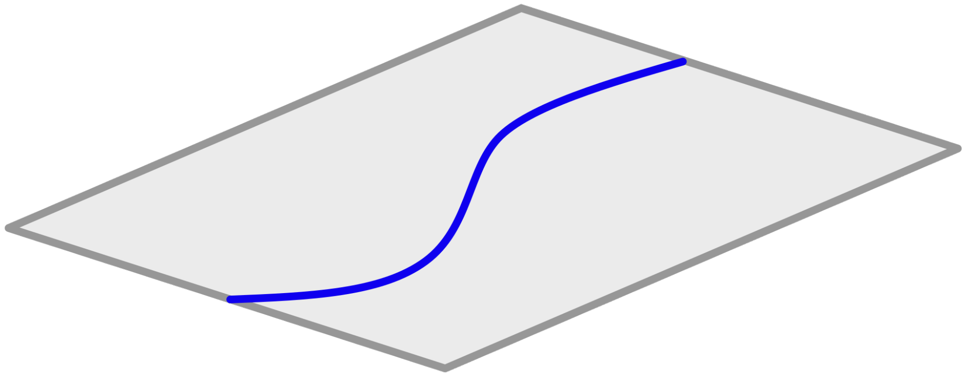
First, let's imagine the figure-eight knot sitting in a plane, except at each intersection, we drop the edge that crosses below downwards (that is, out of the plane). At each crossing, we will then introduce four auxiliary green edges, with orientations specified. Note that one can 'drag' the double-arrow green edge along the knot to match up with the other double-arrow green edge, and likewise for the single-arrow green edge pair.

Figure 4: The figure-eight knot in the plane, with auxiliary green edges added.



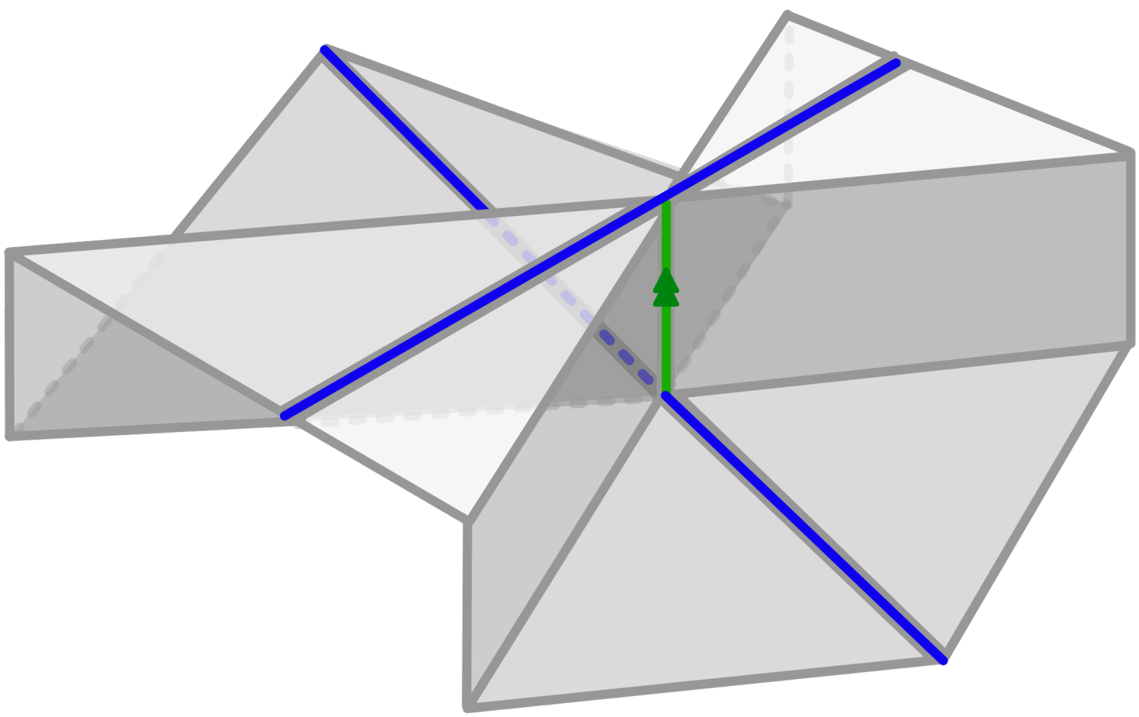
Now imagine there are two balloons: one above the plane and one below the plane, and we inflate them simultaneously. How will the balloons meet? Far away from the knot, they will meet at a plane. But what about regions where the knot exists? Locations on the knot far from crossings are easy, they meet at a plane again (the plane that they meet at is shown in gray).

Figure 5: Far from a crossing, the balloons meet at a plane.



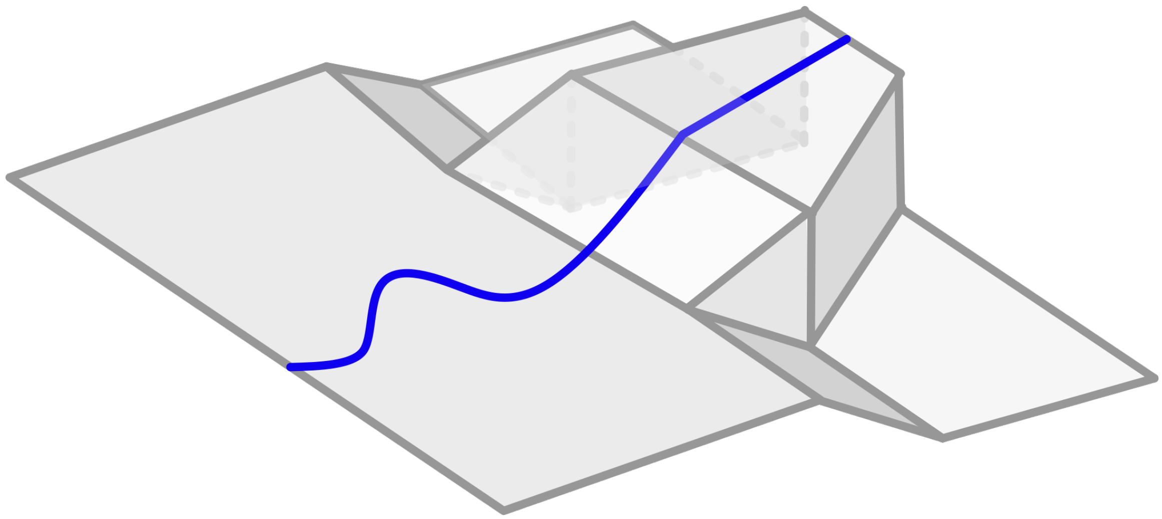
Let's analyze what occurs at a crossing. We'll look specifically at the bottom left crossing in Figure 4 corresponding to the double arrow pointing up (all the other crossings are dealt with similarly). The top balloon will push downwards but want to 'wrap' around the knot, as pictured in Figure 6. The balloon from below pushes upwards. In other words, the balloon on top fills the space above the gray region, and the balloon on the bottom fills the space below.

Figure 6: The planes that the balloons meet at, near a crossing



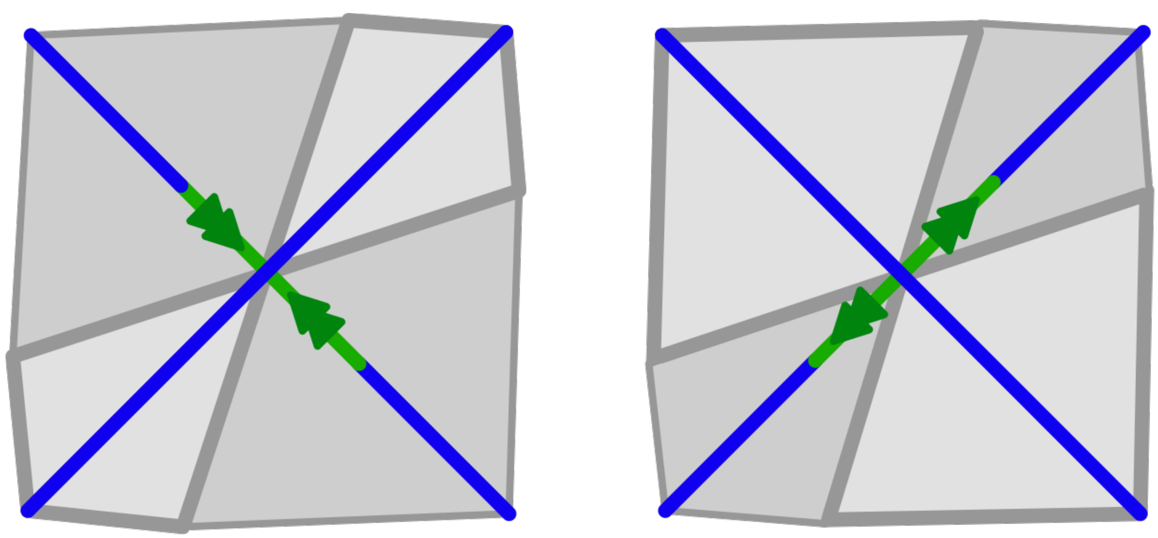
These two figures connect with some smoothing, as seen in Figure 5.

Figure 7: Smoothing between sections of different elevations.



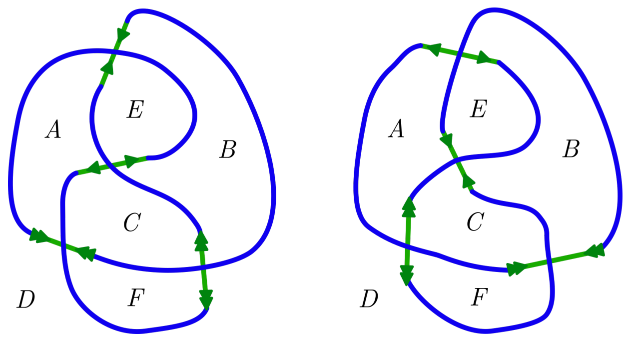
Let's look back at Figure 6. Taking a birds-eye view from above, the top balloon "sees" the top strand of the knot as a line. However, it sees the bottom strand of the knot as a line with the auxiliary green edge in the middle, with the arrows pointing towards the center, as described in the left image of Figure 8. Alternatively, you could imagine that you are on top of this structure, and if you were to feel this crossing with your fingers, you would "feel" the two green arrows pointing towards the center. On the other hand, the bottom balloon "sees" the right image of Figure 8 (and from below, "feels' the two green arrows pointing *away*. Take note that from above (in the left image), it appears as if the strand going from the bottom left to top right passes above. However, taking a view from below, the strand going from top right to bottom left passes "above." Also observe the directions of the arrows: from above, the arrows point inwards, and from below, the arrows point outwards. The lighter color region represents the portion that is closer to the viewer.

Figure 8: Left: What the top balloon "sees," from above. Right: What the bottom balloon "sees," from below.



We'll now apply this same technique to all four crossings. In Figure 9, on the left, we see what the top balloon "sees" throughout the entire knot (from above), and on the right, we see what the bottom balloon "sees" (from below). Note that the crossings swap since we change perspectives. We omit the coloring of gray regions for clarity of crossings, but each of the letters represents a 'face' of the boundary between the balloons (including D, the outer face).

Figure 9: Left: What the top balloon "sees," from above. Right: What the bottom balloon "sees," from below.



We now want to relate our balloon analogy to tetrahedra. We can imagine the top and bottom balloons as two ideal tetrahedra by shrinking the knot to the ideal vertices of the tetrahedra. Remember that we are trying to model the knot complement, and so the knot itself (represented by blue in the previous figures) should not exist in the tetrahedra. So, we retract each blue strand of the knot to a single ideal vertex. This is a valid homeomorphism because we are trying to build a triangulation of the knot *complement*. Considering the ball model of $\mathbb{H}^3$, the knot itself lies on the boundary of the ball, and the complement of a strand on the boundary is homeomorphic to the complement of a point on the boundary (though a point and interval are certainly not homeomorphic!).

Let's turn our attention to what the top balloon 'sees' and shrink the strands to ideal vertices. The first image of Figure 10 shows the result of this process. Each letter represents a 'face' of the boundary between the two balloons (and soon to be two tetrahedra). In the second image, we adjust the edges to be more suggestive of a tetrahedron. Now take a look at the bigons E and F. As noted before, taking a look at Figure 4, the single-arrows edge can be dragged along the knot so that they are identified, and likewise with the double-arrows edge. So we can close the bigons E and F by gluing the identified edges bounding E and F together to obtain the final image in Figure 10, which is an ideal tetrahedron with faces A, B, C, and D.

Figure 10: From the perspective of the top tetrahedron. First: The result of shrinking the knot strands to ideal vertices. Second: Moving some edges around (but keeping the structure the same). Third: Combining the edges to close the bigons F and E.

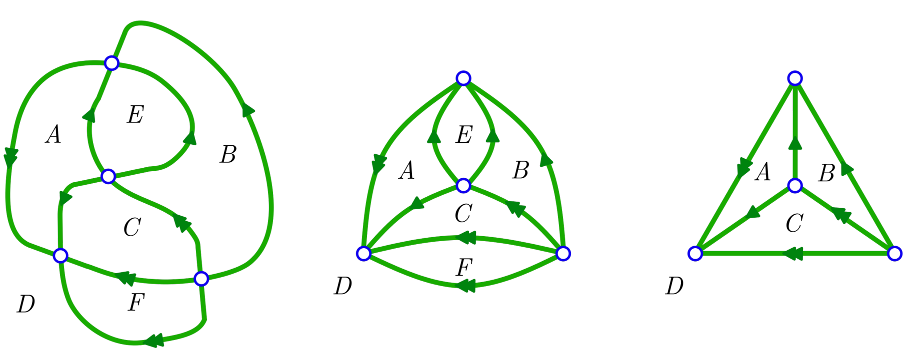
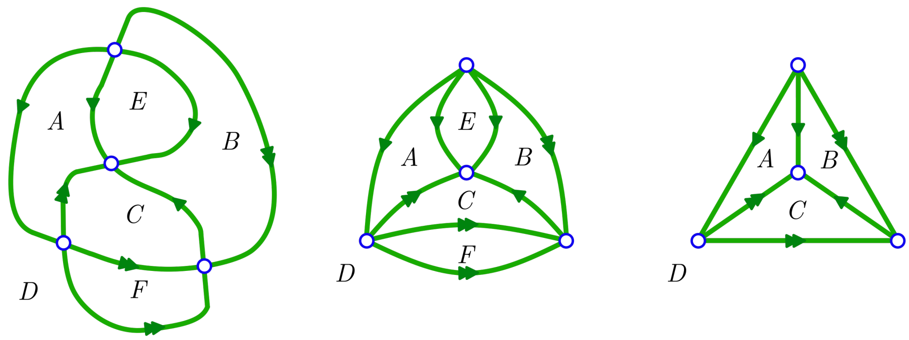


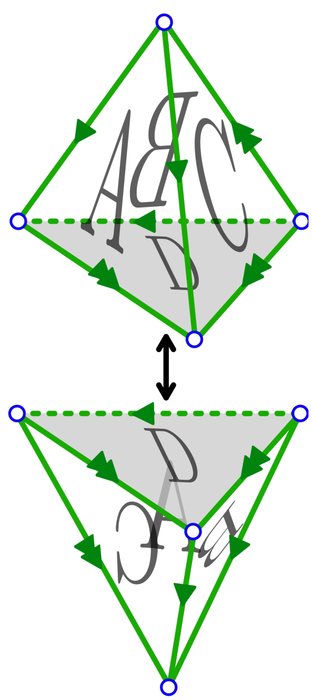
Figure 11: From the perspective of the bottom tetrahedron. First: The result of shrinking the knot strands to ideal vertices. Second: Moving some edges around (but keeping the structure the same). Third: Combining the edges to close the bigons F and E.



As William Menasco once said, we "let bigons be byegones" [Men19].

We repeat the same process to the bottom polyhedron, obtaining a second tetrahedron corresponding to the 'bottom' balloon, as in Figure 11. The final step is to glue/identify each of the four faces together, ensuring that the edges bordering each face are oriented correctly and match together, as described in Figure 12. Thus, we have constructed a decomposition of the figure-8 complement into two ideal tetrahedra, where the knot lives in the ideal vertex, at ∞ (all ideal vertices are identified, which can be checked by vertex chasing). It is of course quite challenging to actually visualize this triangulation, but the construction is complete.

Figure 12: The gluing information for the two tetrahedra.



We now must show the 3-manifold we've constructed admits a hyperbolic structure.

Definition 4.1. A polyhedron P is *regular* if there is an isometry of $\mathbb{H}^3$ which sends any of the vertices of P to any other permutation of the vertices of P.

Let's now finish computing the hyperbolic volume. We can construct a regular ideal tetrahedron by taking a regular tetrahedron in $\mathbb{R}^3$ centered at the origin, such that each of the vertices is on the sphere S^2. Then interpreting this tetrahedron in the ball model of $\mathbb{H}^3$, we obtain a regular *ideal* tetrahedron; any permutation of vertices is realized by an orthogonal transformation of $\mathbb{R}^3$ (i.e., rotations in 3-space), which is an isometry in $\mathbb{H}^3$.

The dihedral angles of this tetrahedron are $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$. Gluing two copies of the regular ideal tetrahedron by the gluing information in Figure 12, we obtain a hyperbolic manifold. This is because by analyzing the gluing we see each edge is glued to 6 other edges. Thus, the angle sum is 2π, which guarantees a hyperbolic manifold, disregarding the second condition mentioned after Theorem 3.1 (which indeed is satisfied). Hence the hyperbolic volume of the knot by Theorem 3.4 is

$$\text{hypvol}(K) = 6\Lambda\left(\frac{\pi}{3}\right) \approx 2.0298832$$

5 Conclusion

5.1 Open Problems

- Though we can calculate the hyperbolic volume to arbitrary precision, is the hyperbolic volume of the knot complement (of any knot) rational or irrational?
- There is a procedure to decompose any hyperbolic knot complement into polyhedra (described in [Pur20]), but naïvely subdividing the polyhedra into tetrahedra may create invalid tetrahedra (they may be flat or oriented the wrong way). Does every hyperbolic knot complement have a triangulation? More generally, does every hyperbolic manifold have a triangulation?

5.2 Further Reading

These resources give a more formal treatment of hyperbolic geometry and their interactions with the topology of knots and manifolds.

- Jessica Purcell, *Hyperbolic Knot Theory*
- William Thurston, *The Geometry and Topology of 3-Manifolds*
- Marc Lackenby, *Lecture Notes on Hyperbolic Manifolds (Hillary 2000)*

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